CREEPING FLOWS

The creeping flow of an incompressible Newtonian fluid satisfies the Stokes equation

$$-\nabla p + \mu \nabla^2 \mathbf{v} + \rho f = 0 \tag{1}$$

and the equation of continuity given as

$$\nabla \cdot \mathbf{v} = 0 \tag{2}$$

Taking divergence of (1) and using (2), in the absence of body force one finds

$$\nabla^2 p = 0 \tag{3}$$

Reciprocity Theorem

Suppose ($\mathbf{v}$ and $\boldsymbol{\tau}$) and $(\mathbf{v}', \boldsymbol{\tau}')$ are two independent solutions (velocity and stress) to (1) and (2). Then

$$\int_S \mathbf{dS} \cdot \boldsymbol{\tau} \cdot \mathbf{v}' = \int_S \mathbf{dS} \cdot \boldsymbol{\tau}' \cdot \mathbf{v}, \tag{4}$$

where S is a closed surface bounding any fluid volume. (See Happel and Brenner, page 85 for the proof).

Minimum Energy Dissipation Theorem (Helmholtz):

The dissipation rate in creeping flow is less than any other incompressible, continuous motion consistent with the same boundary condition. (See Happel and Brenner for the proof.)

Point Force Solution

Consider the response of the Stokes equation to a point force exerted at the origin. Equation (1) and (2) may be rewritten as

$$P_{j,i} = \mu \nabla^2 T_{ij} + \delta_{ij} \delta(\boldsymbol{\tau}) \tag{5}$$

$$T_{ij,i} = 0 \tag{6}$$

The general solution to (5) and (6) in an unbounded domain is given as

$$T_{ij} = \frac{1}{8\pi\mu r}\left[\delta_{ij} + \frac{r_i r_j}{r^2}\right], \ P_i = \frac{r_i}{4\pi r^3}. \tag{7}$$

Here, T_{ij} is referred to as the Oseen tensor. The solution in a two-dimensional flow is given by

$$T_{ij} = \frac{1}{4\pi\mu}\left[-\delta_{ij}\ln|r| + \frac{r_i r_j}{r^2}\right], \ P_i = \frac{r_i}{2\pi r^2} \tag{8}$$

Using the Green theorem, the velocity vector may be represented as

$$v_i = \int_V T_{ij}(\mathbf{r}-\mathbf{r}')\rho f(\mathbf{r}')dV + \int_S [\tau_{jk}(\mathbf{r}')T_{ki}(\mathbf{r}-\mathbf{r}') - R_{ijk}(\mathbf{r}-\mathbf{r}')v_k(\mathbf{r}')]dS_j, \tag{9}$$

where

$$R_{ijk}(\mathbf{r}) = \frac{3r_i r_j r_k}{4\pi r^5} \tag{10}$$

and

$$p(\mathbf{r}) = \int_V P_i(\mathbf{r}-\mathbf{r}')\rho f_i(\mathbf{r}')dV + \int_S [\tau_{ij}(\mathbf{r}')P_j(\mathbf{r}-\mathbf{r}') + 2\mu v_j(\mathbf{r}')P_{j,i}(\mathbf{r}-\mathbf{r}')]dS_i \tag{11}$$

Equations (9) and (11) give the velocity and the pressure fields for arbitrary given distribution of stress and velocity on solid surface boundaries.

For an unbounded flow and for a point force $\rho f_i = F_i \delta(\mathbf{r})$, from (9) and (11) it follows that

$$v_i = F_j T_{ij}(r), \ \mathbf{v} = \frac{\mathbf{F}}{8\pi\mu r}\cdot\left[\mathbf{I} + \frac{\mathbf{rr}}{r^2}\right], \tag{12}$$

$$p(r) = F_i P_i(r), \ p = \frac{\mathbf{F}\cdot\mathbf{r}}{4\pi r^3} \tag{13}$$

which is referred to as Stokeslet.

General Solution in Spherical Coordinate

In the absence of body force the general solution for creeping motion in spherical coordinates was given by Lamb. From (3) it follows that

$$p = \sum_{n=0}^{\infty} P_n \ , \ P_n = \left(A_n r^n + B_n r^{-n-1}\right) Y_{nm}(\theta,\phi), \tag{14}$$

where

$$Y_{mn}(\theta,\phi) = \left(C_{mn}\cos m\phi + D_{mn}\sin m\phi\right) P_n^m(\cos\theta) \tag{15}$$

with

$$m = 0,1,2,\ldots,n \ \text{ and } \ n = 0,1,2,\ldots$$

are spherical harmonics. The corresponding expression for the velocity field is given as

$$v = \sum_{n=0}^{\infty}\left[\nabla\times\left(r\xi_n\right) + \nabla\left(\Phi_n\right) + \frac{(n+3)r^2\nabla P_n - 2n\mathbf{r}P_n}{2\mu(n+1)(2n+3)}\right] \tag{16}$$

where ξ_n and Φ_n are also spherical harmonics similar to those given by (14).

Expressions for the stress field and the corresponding drag and torque were given by Happel and Brenner (pages 66-67).

Faxen's Law

When the fluid motion at infinity is nonuniform, the expression for the force and torque on the sphere becomes

$$\mathbf{F} = 6\pi\mu R\left[\left(\mathbf{v}_\infty|_o - \mathbf{U}\right) + \frac{1}{6}R^2\nabla^2\mathbf{v}_\infty|_o\right], \tag{17}$$

$$\mathbf{T_0} = 8\pi\mu R^3\left(\frac{1}{2}\nabla\times\mathbf{v}_\infty|_o - \boldsymbol{\omega}\right), \tag{18}$$

where $\mathbf{U}$ is the translational velocity of the sphere, $\boldsymbol{\omega}$ is its spin, R is its radius, and $\mathbf{v}_\infty$ is the velocity field far away. The corresponding stresslet is given by

$$\mathbf{S} = \frac{20}{3}\pi R^3\mu\left(1 + \frac{R^2}{10}\nabla^2\right)\mathbf{d}\Big|_o \tag{19}$$