

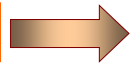
Incompressible Viscous Flows

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- ▶ Navier-Stokes Equation
- ▶ Vorticity Transport Equation
- ▶ Plane Flows and Stream Function
- ▶ Cylindrical Flows
- ▶ Spherical Coordinate Systems
- ▶ Plane Stagnation Flows
- ▶ Axisymmetric Stagnation Flows

Continuity

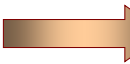


$$\nabla \cdot \mathbf{v} = 0$$

Navier-Stokes

$$\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{v} = -\frac{1}{\rho} \nabla P + \nu \nabla^2 \mathbf{v}$$

Vorticity



$$\boldsymbol{\omega} = \nabla \times \mathbf{v}$$

Vorticity Transport Equation

$$\frac{\partial \boldsymbol{\omega}}{\partial t} - \nabla \times (\mathbf{v} \times \boldsymbol{\omega}) = -\nu \nabla \times \nabla \times \boldsymbol{\omega}$$

Vorticity Transport Equation Clarkson University

Vector Identity

$$\nabla \times (\mathbf{v} \times \boldsymbol{\omega}) = (\nabla \cdot \boldsymbol{\omega})\mathbf{v} - (\nabla \cdot \mathbf{v})\boldsymbol{\omega} + \boldsymbol{\omega} \cdot \nabla \mathbf{v} - \mathbf{v} \cdot \nabla \boldsymbol{\omega}$$

Vorticity Transport Equation

$$\frac{\partial \boldsymbol{\omega}}{\partial t} + \mathbf{v} \cdot \nabla \boldsymbol{\omega} = \boldsymbol{\omega} \cdot \nabla \mathbf{v} + \mathbf{v} \nabla^2 \boldsymbol{\omega}$$

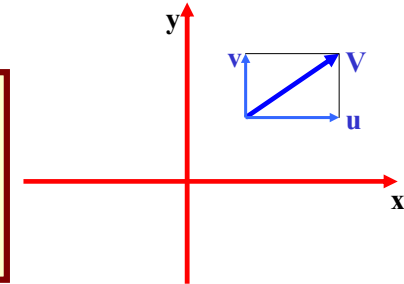
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Stream Function

$$\mathbf{v} = \nabla \times (\mathbf{k}\psi) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 0 & 0 & \psi \end{vmatrix} = \frac{\partial \psi}{\partial y} \mathbf{i} - \frac{\partial \psi}{\partial x} \mathbf{j}$$



$$u = \frac{\partial \psi}{\partial y}$$

$$v = -\frac{\partial \psi}{\partial x}$$

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Vorticity



$$\omega_z = \boldsymbol{\omega} = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = -\nabla^2 \psi$$

Vorticity Transport Equation

$$\frac{\partial \omega}{\partial t} + u \frac{\partial \omega}{\partial x} + v \frac{\partial \omega}{\partial y} = \nu \nabla^4 \omega$$

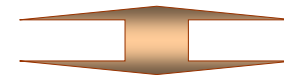
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Vorticity Transport Equation in terms of Stream Function

$$\frac{\partial}{\partial t} \nabla^2 \psi + \frac{\partial \psi}{\partial y} \frac{\partial}{\partial x} \nabla^2 \psi - \frac{\partial \psi}{\partial x} \frac{\partial}{\partial y} \nabla^2 \psi = \nu \nabla^4 \psi$$



$$\frac{\partial}{\partial t} \nabla^2 \psi + \frac{\partial(\nabla^2 \psi, \psi)}{\partial(x, y)} = \nu \nabla^4 \psi$$

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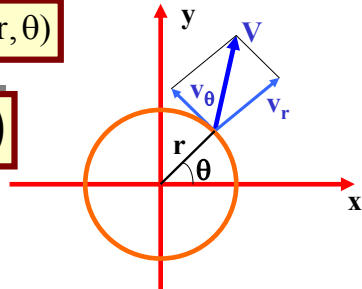
$$v_z = 0 \quad v_r(r, \theta) \quad v_\theta(r, \theta)$$

$$\mathbf{v} = \nabla \times (\mathbf{e}_z \psi(r, \theta))$$

$$\mathbf{v} = \frac{1}{r} \frac{\partial \psi}{\partial \theta} \mathbf{e}_r - \frac{\partial \psi}{\partial r} \mathbf{e}_\theta$$

$$v_r = \frac{1}{r} \frac{\partial \psi}{\partial \theta}$$

$$v_\theta = -\frac{\partial \psi}{\partial r}$$



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Vorticity →

$$\omega_z = \omega = -\nabla^2 \psi$$

$$\nabla^2 \psi = \frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial \psi}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 \psi}{\partial \theta^2}$$

Vorticity Transport Equation

$$\frac{\partial \omega}{\partial t} + v_r \frac{\partial \omega}{\partial r} + v_\theta \frac{1}{r} \frac{\partial \omega}{\partial \theta} = \nu \nabla^2 \omega$$

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Stream Function Equation

$$\frac{\partial}{\partial t} \nabla^2 \psi + \frac{1}{r} \frac{\partial \psi}{\partial \theta} \frac{\partial}{\partial r} \nabla^2 \psi - \frac{\partial \psi}{\partial r} \frac{1}{r} \frac{\partial}{\partial \theta} \nabla^2 \psi = \nu \nabla^4 \psi$$

$$\frac{\partial}{\partial t} \nabla^2 \psi + \frac{1}{r} \frac{\partial (\nabla^2 \psi, \psi)}{\partial (r, \theta)} = \nu \nabla^4 \psi$$

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Axisymmetric Flows Clarkson University

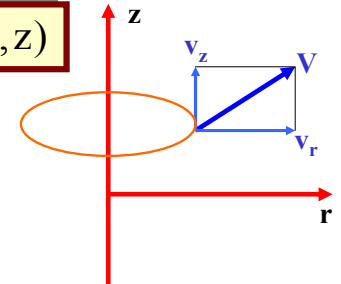
$$v_\theta = 0 \quad v_r(r, z) \quad v_z(r, z)$$

$$\mathbf{v} = \nabla \times \left(\mathbf{e}_\theta \frac{\psi}{r}(r, z) \right)$$

$$\mathbf{v} = -\frac{1}{r} \frac{\partial \psi}{\partial z} \mathbf{e}_r + \frac{1}{r} \frac{\partial \psi}{\partial r} \mathbf{e}_z$$

$$v_z = \frac{1}{r} \frac{\partial \psi}{\partial r}$$

$$v_r = -\frac{1}{r} \frac{\partial \psi}{\partial z}$$



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Vorticity

$$\boldsymbol{\omega} = \nabla \times \mathbf{v} = \mathbf{e}_\theta \left(\frac{\partial v_r}{\partial z} - \frac{\partial v_z}{\partial r} \right)$$

$$\boldsymbol{\omega} = -\mathbf{e}_\theta \frac{1}{r} \left(\frac{\partial^2 \psi}{\partial z^2} + r \frac{\partial}{\partial r} \left(\frac{1}{r} \frac{\partial \psi}{\partial r} \right) \right)$$

$$\omega_\theta = \omega = -\frac{1}{r} E^2 \psi$$

$$E^2 \psi = \frac{\partial^2 \psi}{\partial r^2} - \frac{1}{r} \frac{\partial \psi}{\partial r} + \frac{\partial^2 \psi}{\partial z^2}$$

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Vorticity Transport Equation

$$\frac{\partial \omega_\theta}{\partial t} + v_r \frac{\partial \omega_\theta}{\partial r} + v_z \frac{\partial \omega_\theta}{\partial z} - \frac{v_r \omega_\theta}{r} = -v \nabla \times \nabla \times (\omega_\theta \mathbf{e}_\theta) |_{\theta\text{-comp.}}$$

$$\frac{\partial}{\partial t} (E^2 \psi) - \frac{1}{r} \frac{\partial \psi}{\partial z} \frac{\partial}{\partial r} (E^2 \psi) + \frac{1}{r} \frac{\partial \psi}{\partial r} \frac{\partial}{\partial z} (E^2 \psi) + \frac{2}{r^2} \frac{\partial \psi}{\partial z} E^2 \psi = v E^4 \psi$$

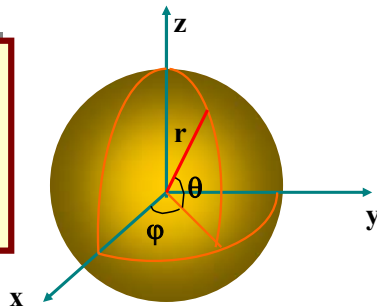
$$\frac{\partial}{\partial t} (E^2 \psi) - \frac{1}{r} \frac{\partial (E^2 \psi, \psi)}{\partial (r, z)} + \frac{2}{r^2} \frac{\partial \psi}{\partial z} E^2 \psi = v E^4 \psi$$

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$$\begin{cases} x = r \cos \theta \cos \varphi \\ y = r \cos \theta \sin \varphi \\ z = r \sin \theta \end{cases}$$



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$$v_\varphi = 0$$

$$v_r(r, \theta)$$

$$v_\theta(r, \theta)$$

Stream Function

$$\mathbf{v} = \nabla \times \left(\frac{\mathbf{e}_\varphi \psi(r, \theta)}{r \sin \theta} \right)$$

$$\mathbf{v} = \frac{1}{r^2 \sin \theta} \frac{\partial \psi}{\partial \theta} \mathbf{e}_r - \frac{1}{r \sin \theta} \frac{\partial \psi}{\partial r} \mathbf{e}_\theta$$

$$v_r = \frac{1}{r^2 \sin \theta} \frac{\partial \psi}{\partial \theta}$$

$$v_\theta = -\frac{1}{r \sin \theta} \frac{\partial \psi}{\partial r}$$

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Vorticity Transport Equation

$$\frac{\partial}{\partial t} (E^2 \psi) + \frac{1}{r^2 \sin \theta} \frac{\partial (E^2 \psi, \psi)}{\partial (r, \theta)} + \frac{2E^2 \psi}{r^2 \sin^2 \theta} \left(\frac{\partial \psi}{\partial r} \cos \theta - \frac{1}{r} \frac{\partial \psi}{\partial \theta} \sin \theta \right) = \nu E^4 \psi$$

$$E^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2} - \frac{\cot \theta}{r^2} \frac{\partial}{\partial \theta}$$

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Intrinsic Coordinates Clarkson University

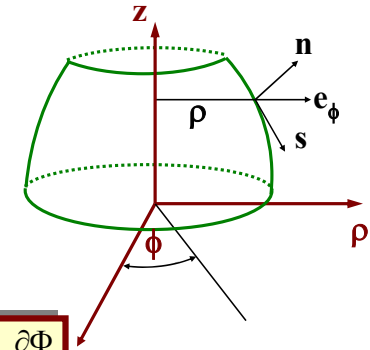
Metric Coefficients

$$h_1 = h_n = 1$$

$$h_2 = h_\phi = \rho$$

$$h_3 = h_s = 1$$

$$\nabla \Phi = \mathbf{n} \frac{\partial \Phi}{\partial n} + \mathbf{s} \frac{\partial \Phi}{\partial s} + \mathbf{e}_\phi \frac{1}{\rho} \frac{\partial \Phi}{\partial \phi}$$



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Stream Function

$$\mathbf{v} = \nabla \times \left(\mathbf{e}_\phi \frac{\psi(s, n)}{\rho} \right) = -\frac{1}{\rho} \frac{\partial \psi}{\partial s} \mathbf{n} + \frac{1}{\rho} \frac{\partial \psi}{\partial n} \mathbf{s}$$

$$v_n = -\frac{1}{\rho} \frac{\partial \psi}{\partial s}$$

$$v_s = \frac{1}{\rho} \frac{\partial \psi}{\partial n}$$

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Intrinsic Axisymmetric Flows Clarkson University

Vorticity

$$\boldsymbol{\omega} = \nabla \times \mathbf{v} = -\mathbf{e}_\phi \left[\frac{\partial}{\partial n} \left(\frac{1}{\rho} \frac{\partial \psi}{\partial n} \right) + \frac{\partial}{\partial s} \left(\frac{1}{\rho} \frac{\partial \psi}{\partial s} \right) \right] = -\frac{1}{\rho} E^2 \psi$$

$$E^2 = \rho \left[\frac{\partial}{\partial n} \left(\frac{1}{\rho} \frac{\partial}{\partial n} \right) + \frac{\partial}{\partial s} \left(\frac{1}{\rho} \frac{\partial}{\partial s} \right) \right]$$

$$E^2 = \frac{\partial^2}{\partial \rho^2} - \frac{1}{\rho} \frac{\partial}{\partial \rho} + \frac{\partial^2}{\partial z^2}$$

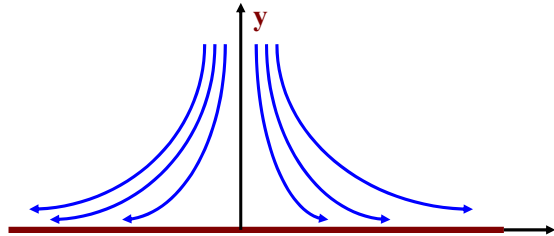
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Plane Stagnation Flows Clarkson University

Navier-Stokes Equation

$$\frac{\partial \psi}{\partial y} \frac{\partial}{\partial x} \nabla^2 \psi - \frac{\partial \psi}{\partial x} \frac{\partial}{\partial y} \nabla^2 \psi = \nu \nabla^4 \psi$$



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Potential Flow

$$\psi = axy$$

$$U = ax$$

$$V = -ay$$

Viscous Flow

$$\psi = xf(y)$$

$$u = xf'$$

$$v = f$$

$$\nabla^2 \psi = xf''$$

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Navier-Stokes

$$f'f'' - ff''' = \nu f^{(4)}$$

Integrating

$$f'^2 - ff'' = \nu f''' + c$$

Boundary Conditions

$$y = 0$$

$$u = 0$$

$$v = 0$$

$$y \rightarrow \infty$$

$$\psi = axy$$

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Boundary Conditions

$$f(0) = f'(0) = 0$$

$$y \rightarrow \infty$$

$$f \rightarrow ay$$

Navier-Stokes

$$f'^2 - ff'' = \nu f''' + a^2$$

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Changing Variables

$$\eta = \sqrt{\frac{a}{\nu}} y \quad f = \sqrt{a\nu} \phi(\eta)$$

Navier-Stokes

$$\phi''' + \phi\phi'' - \phi'^2 + 1 = 0$$

Boundary Conditions

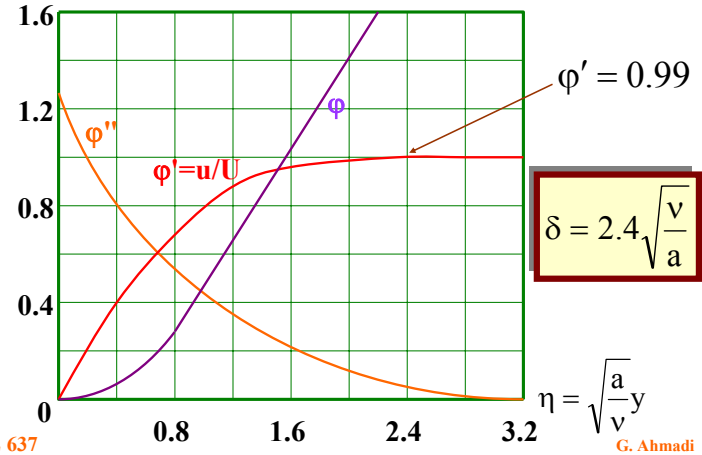
$$\phi(0) = \phi'(0) = 0$$

$$\eta \rightarrow \infty \quad \phi' = 1$$

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Concluding Remarks Clarkson University

- Navier-Stokes Equation for Viscous Flows
- Vorticity Transport Equation
- Plane Flows and Stream Function
- Cylindrical Spherical Coordinate Systems
- Stagnation Flows

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Thank you!

Questions?

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