INDICIAL NOTATION (Cartesian Tensor)

Basic Rules

i) A free index appears only once in each term of a tensor equation. The equation then holds for all possible values of that index.
ii) Summation is implied on an index, which appears twice.
iii) No index can appear more than twice in any term.

Definition (Cartesian Tensors)

Consider a change of frame

$$x_i^* = Q_{ij}x_j,\ x_j = Q_{ij}x_i^*,\ \det|Q_{ij}| = \pm 1,\ Q_{ij}Q_{ik} = \delta_{jk},\ Q_{ij}Q_{kj} = \delta_{ik}$$

The quantities $T(\mathbf{x})$, $\mathbf{v}(\mathbf{x})$, and $\mathbf{t}(\mathbf{x})$ are said to be a scalar, a vector, or a second order tensor if they transform according to the rules:

$$T^* = T,\ \mathbf{v}^* = \mathbf{Q}\cdot\mathbf{v},\ \boldsymbol{\tau}^* = \mathbf{Q}\cdot\mathbf{t}\cdot\mathbf{Q}^T$$

or in component forms:

$$v_i^* = Q_{ij}v_j,\ t_{ij}^* = Q_{ik}Q_{jl}t_{kl}$$

Similarly, a third order tensor $\boldsymbol{\lambda}$ transforms as:

$$\lambda_{ijk}^* = Q_{im}Q_{jn}Q_{kl}\lambda_{mnl}$$

Tensor Operations

Gradient:

$$(\nabla\varphi)_i = \frac{\partial\varphi}{\partial x_i} = \varphi_{,i}$$

$$(\nabla\mathbf{v})_{ij} = \frac{\partial v_j}{\partial x_\varepsilon} = v_{j,i}$$

Divergence:

$$\nabla\cdot\mathbf{v} = v_{i,i}$$

$$(\nabla\cdot\boldsymbol{\tau})_j = \frac{\partial\tau_{ij}}{\partial x_i} = \tau_{ij,i}$$

Curl:

$$(\nabla \times \mathbf{U})_i = \varepsilon_{ijk} \frac{\partial U_k}{\partial x_j} = \varepsilon_{ijk} U_{k,j}$$

Here, ε_{ijk} is the permutation symbol. The permutation symbol is used for evaluating the determinant. e.g.,

$$\det|\mathbf{A}| = \varepsilon_{ijk} A_{1i} A_{2j} A_{3k} ,$$

The inner product of the permutation symbol satisfy the following identity:

$$\varepsilon_{ijk}\varepsilon_{imn} = \delta_{jm}\delta_{kn} - \delta_{jn}\delta_{km}$$

Laplacian:

$$\nabla^2 \varphi = \frac{\partial^2 \varphi}{\partial x_i \partial x_i} = \varphi_{,ii}$$

Isotropic Tensors

Isotropic tensors are tensors, which are form invariant under all possible rotations of the frame of reference. The most general forms of the isotropic tensors are:

Rank zero: All scalars isotropic
Rank one: None
Rank two: $\alpha\delta_{ij}$, α a scalar and $\delta_{ij} = \text{Kronecker delta}$
Rank three: $\alpha\varepsilon_{ijk}$
Rank four: $\alpha\delta_{ij}\delta_{kl} + \beta(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}) + \gamma(\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk})$