Incompressible Viscous Flows

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Outline

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- Navier-Stokes Equation
- Steady Parallel Flows
- Boundary Conditions
- Unsteady Parallel Flow
- Boundary Layer Flows
- Flow over a Flat Plate
- Blasius Solution
- Laminar Boundary Layer

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Navier-Stokes Equation

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Continuity

$$\nabla \cdot \mathbf{v} = 0$$

Navier-Stokes

$$\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{v} = -\frac{1}{\rho} \nabla P + \nu \nabla^2 \mathbf{v}$$

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Navier-Stokes Equation

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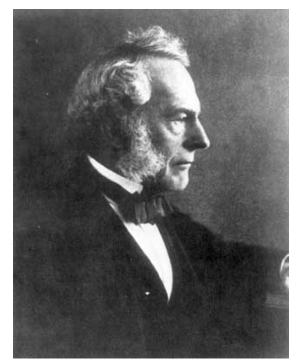
Navier



Stokes

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Navier-Stokes Equation

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Incompressible Fluid

$$\rho\left(\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+w\frac{\partial u}{\partial z}\right)=\rho g_x-\frac{\partial p}{\partial x}+\mu\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}\right)$$

$$\rho\left(\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}+w\frac{\partial v}{\partial z}\right)=\rho g_y-\frac{\partial p}{\partial y}+\mu\left(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}+\frac{\partial^2 v}{\partial z^2}\right)$$

$$\rho\left(\frac{\partial w}{\partial t}+u\frac{\partial w}{\partial x}+v\frac{\partial w}{\partial y}+w\frac{\partial w}{\partial z}\right)=\rho g_z-\frac{\partial p}{\partial z}+\mu\left(\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}+\frac{\partial^2 w}{\partial z^2}\right)$$

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}=0$$

4 Equations for 4 unknowns u,v,w and p

Steady Parallel Flows

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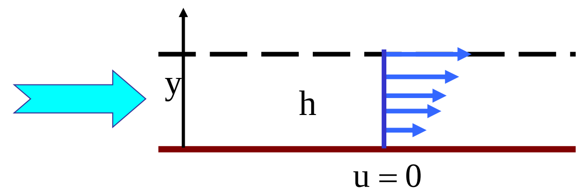
Cartesian Coordinates



$$u(y),\ v=0,\ w=0$$

Steady Parallel Flows

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Cartesian Coordinates

$$\rho g_x-\frac{\partial p}{\partial x}+\mu\frac{d^2u}{dy^2}=0$$

General Solution

$$u=-\frac{1}{2\mu}\left(\rho g_x-\frac{\partial p}{\partial x}\right)y^2+Ay+B$$

Boundary Conditions

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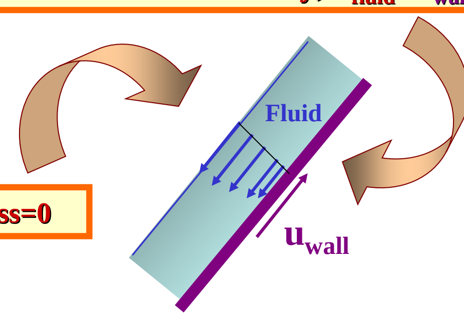
Solid Walls

A viscous fluid sticks to its boundary, $u_{fluid}=u_{wall}$

Free Surface

Shear Stress=0



Boundary Conditions

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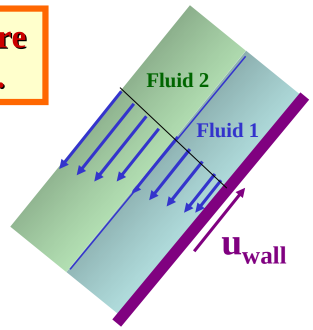
Between Two Fluids

Velocity and shear stress are the same at the interface.

$u_1 = u_2$

$\tau_1 = \tau_2$



Steady Parallel Flows

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Cartesian

$$\rho g_x - \frac{\partial p}{\partial x} + \mu \frac{d^2 u}{dy^2} = 0$$

$$u = -\frac{1}{2\mu}(\rho g_x - \frac{\partial p}{\partial x})y^2 + Ay + B$$

Cylindrical Axial

$$\rho g_z - \frac{\partial p}{\partial z} + \mu \frac{1}{r}\frac{d}{dr}(r\frac{dv_z}{dr}) = 0$$

$$v_z = -\frac{1}{4\mu}(\rho g_x - \frac{\partial p}{\partial z})r^2 + A \ln r + B$$

Rotating

$$\frac{d^2 v_\theta}{dr^2} + \frac{1}{r}\frac{dv_\theta}{dr} - \frac{v_\theta}{r^2} = 0$$

$$v_\theta = Ar + \frac{B}{r}$$

$$\frac{dp}{dr} = \rho \frac{v_\theta^2}{r}$$

Plate Suddenly Set in Motion

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$$\frac{\partial u}{\partial t} = \nu \frac{\partial^2 u}{\partial y^2}$$

B.C.

$y = 0 \Rightarrow u = U_0$

$y = \infty \Rightarrow u = 0$

I.C.

$t = 0 \Rightarrow u = 0$

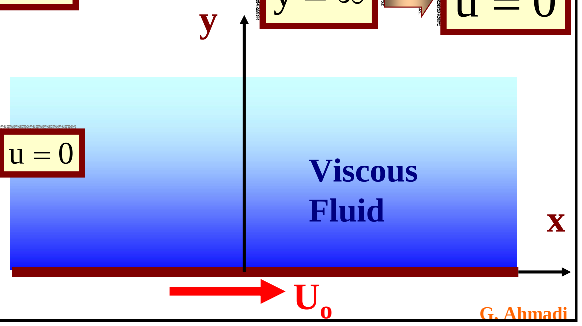

Plate Suddenly Set in Motion

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Similarity Solution

Let $t \sim t^1$ $y \sim t^a$

Navier-Stokes $\Rightarrow 1 = 2a \Rightarrow a = 1/2$

Similarity Variables $\Rightarrow$ $\eta = \frac{y}{2\sqrt{\nu t}}$ $\quad \frac{u}{U_0} = f(\eta)$

Plate Suddenly Set in Motion

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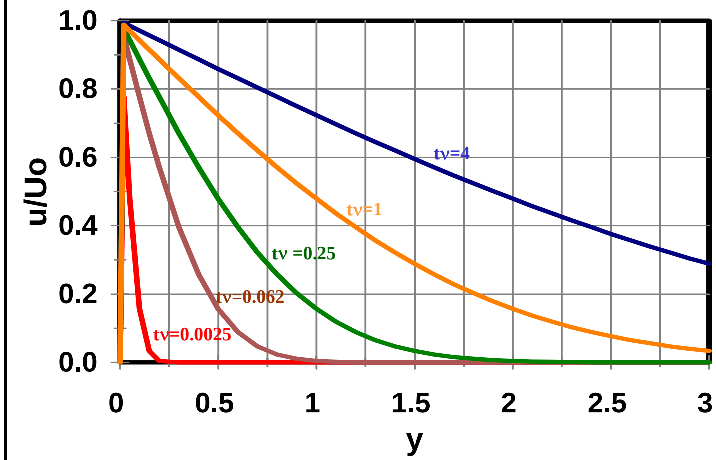
NS → $f'' + 2\eta f' = 0$

→ $f' = c e^{-\eta^2}$

B.C. → $f(0) = 1$, $f(\infty) = 0$

$$f = 1 - \frac{2}{\sqrt{\pi}} \int_0^{\eta} e^{-\eta_1^2} d\eta_1 = 1 - \operatorname{erf}(\eta)$$

Solution → $u = U_0 \operatorname{erfc}\left(\frac{y}{2\sqrt{\nu t}}\right)$



Variation of velocity profile with time.

Boundary Layer

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Boundary Layer over a Flat Plate

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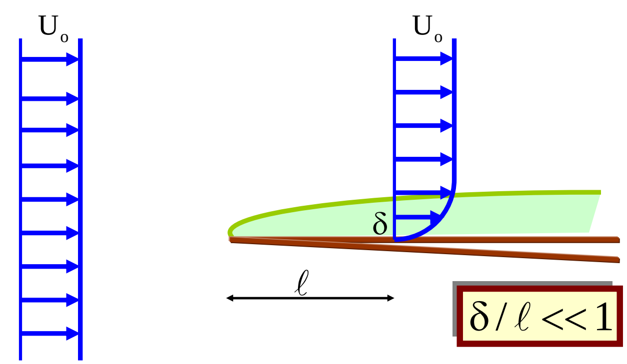

$\delta / \ell << 1$

Laminar Boundary Layer

Boundary Layer Thickness

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Boundary Layer Thickness, δ → Distance at which $\frac{u}{U_0} = 0.99$

Displacement Thickness → $\delta^* = \int_0^\infty \left(1 - \frac{u}{U_0}\right) dy$

Momentum Thickness → $\theta = \int_0^\infty \frac{u}{U_0}\left(1 - \frac{u}{U_0}\right) dy$

Shape Factor → $H = \frac{\delta^*}{\theta}$

Boundary Layer Theory

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Steady Two-D Flows

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)$$

$$u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial y} + \nu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right)$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

Boundary Layer Equations

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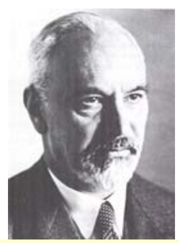
$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{dp}{dx} + \nu\frac{\partial^2 u}{\partial y^2}$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$



Ludwig Prandtl

Boundary Conditions →

$$\text{at } y = 0 \quad u = 0,\ v = 0$$
$$\text{at } y = \infty \quad u = U_0$$

Boundary Layer over a Flat Plate

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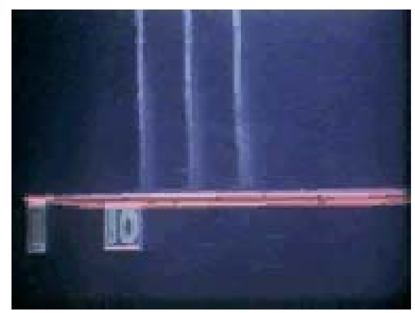

Boundary Layer over a Flat Plate

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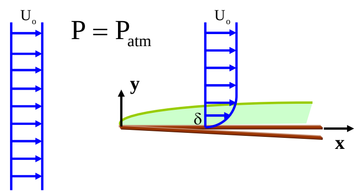
$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=\nu\frac{\partial^2 u}{\partial y^2}$$

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0$$



Boundary Conditions

$$\text{at } y=0 \quad u=0,\ v=0$$
$$\text{at } y=\infty \quad u=U_o$$

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Blasius Solution

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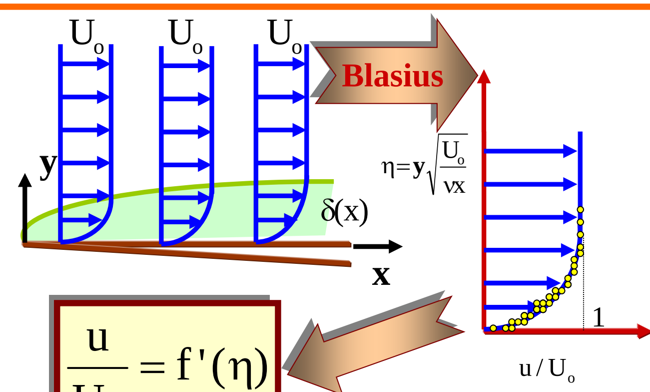

$$\frac{u}{U_o}=f'(\eta)$$

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Blasius Similarity Solution

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$$\eta=y\sqrt{\frac{U_o}{\nu x}}$$ $$\frac{u}{U_o}=f'(\eta)$$ $$\frac{\partial u}{\partial y}=f''(\eta)\sqrt{\frac{U_o}{\nu x}}$$

Blasius Equation

Boundary Layer Eq.

$$ff''+2f'''=0$$

Boundary Conditions

$$\text{at } \eta=0 \quad f=0,\ f'=0$$
$$\text{at } \eta=\infty \quad f'=1$$

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Blasius Similarity Solution

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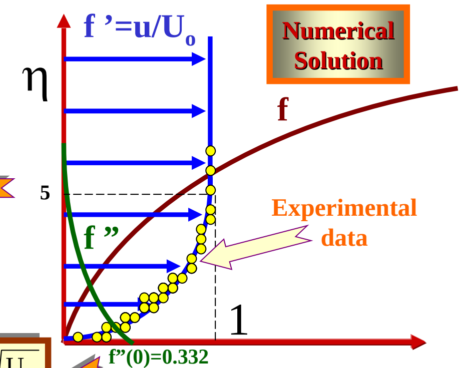
Boundary Layer Thickness, δ

$$\delta=5\sqrt{\frac{\nu x}{U_o}}$$

$$\frac{\delta}{x}=5\,Re_x^{-\frac{1}{2}}$$

$$\tau=\mu\left.\frac{du}{dy}\right|_{y=0}=\mu U_o f''(0)\sqrt{\frac{U_o}{\nu x}}$$



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Boundary Layer over a Flat Plate

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Friction Coefficient

$$C_F = \frac{\tau}{\frac{1}{2}\rho U_o^2} = \frac{2f''(0)}{\sqrt{R_{ex}}} = \frac{0.664}{\sqrt{R_{ex}}}$$

Drag Coefficient

$$C_D = \frac{D}{\frac{1}{2}\rho U_o^2 \ell} = \frac{4f''(0)}{\sqrt{R_{e\ell}}} = \frac{1.328}{\sqrt{R_{e\ell}}}$$

Boundary Layer over a Flat Plate

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Displacement Thickness

$$\delta^* = \int_0^\infty \left(1 - \frac{U}{U_0}\right) dy = 1.721\sqrt{\frac{\nu x}{U_0}}$$

Momentum Thickness

$$\theta = \int_0^\infty \frac{U}{U_0}\left(1 - \frac{U}{U_0}\right) dy = 0.664\sqrt{\frac{\nu x}{U_0}}$$

Shape Factor

$$H = \frac{\delta^*}{\theta} = 2.51$$

Summary

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Thank you!

Questions?