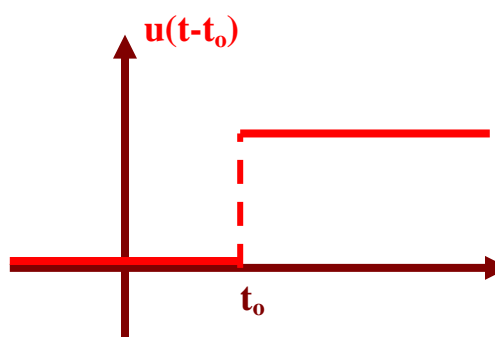


## Review of Engineering Mathematics

### Special Functions

#### Unit step function

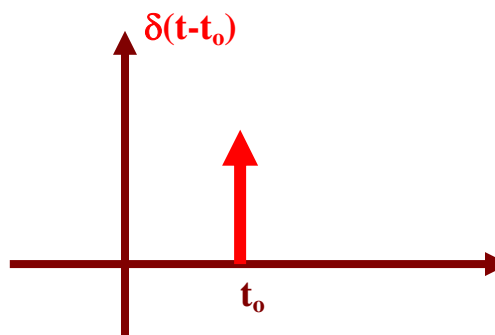
$$u(t - t_0) = \begin{cases} 1 & t \geq t_0 \\ 0 & t < t_0 \end{cases}$$



#### Delta Function

Dirac delta function is defined as

$$\delta(t - t_0) = \frac{du(t - t_0)}{dt}$$



Note that

$$\int_{-\infty}^{+\infty} \delta(t - t_0) dt = \int_{t_0 - \varepsilon}^{t_0 + \varepsilon} \delta(t - t_0) dt = 1$$

Also

$$\int_{-\infty}^{+\infty} f(t) \delta(t - t_0) dt = \int_{t_0 - \varepsilon}^{t_0 + \varepsilon} f(t) \delta(t - t_0) dt = f(t_0)$$

$$\int_{-\infty}^t f(t_1) \delta(t_1 - t_0) dt_1 = f(t_0) u(t - t_0)$$

$$\delta[a(t - t_0)] = \frac{1}{|a|} \delta(t - t_0) \quad \text{for } a \neq 0$$

### Error Function

$$\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt, \quad \operatorname{erfc}(x) = 1 - \operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_x^\infty e^{-t^2} dt$$

$$\operatorname{erf}(0) = 0 = \operatorname{erfc}(\infty), \quad \operatorname{erf}(-x) = -\operatorname{erf}(x)$$

### Exponential Integrals

$$E_n(x) = \int_1^\infty \frac{e^{-xt}}{t^n} dt, \quad E_i(x) = \int_{-\infty}^x \frac{e^t}{t} dt$$

## Differential Equations

### Linear First-Order Differential Equations

$$\frac{dy}{dx} + P(x)y = Q(x)$$

$$y = ce^{-\int_0^x P(x_1) dx_1} + \int_0^x e^{-\int_{x_1}^x P(x_1) dx_2} Q(x_1) dx_1$$

#### Example

$$\frac{dy}{dx} + by = Q(x), \quad \text{with } y(0) = 0$$

$$y = \int_0^x e^{-b(x-x_1)} Q(x_1) dx_1$$

### Second-Order Differential Equations with Constant Coefficients

$$\frac{d^2y}{dx^2} + a \frac{dy}{dx} + by = 0$$

Form characteristic equation

$$m^2 + am + b = 0 \quad \text{Solve for } \rightarrow \quad m_1, m_2$$

Different cases:

- i.  $m_1, m_2$  are real and  $m_2 \neq m_1$ , then

$$y = c_1 e^{m_1 x} + c_2 e^{m_2 x}$$

- ii.  $m_2 = m_1 = m \sim$  real, then

$$y = e^{mx} (c_1 + c_2 x)$$

- iii.  $m_1 = p + qi, \quad m_2 = p - qi$ , where  $p = -\frac{a}{2}, \quad q = \sqrt{b - \frac{a^2}{4}}$ , then

$$y = e^{px} (c_1 \cos qx + c_2 \sin qx)$$

### Nonhomogeneous Second-Order Differential Equations (Particular Solutions)

$$\frac{d^2 y}{dx^2} + a \frac{dy}{dx} + by = R(x)$$

- i.  $y_p = \frac{e^{m_1 x}}{m_1 - m_2} \int e^{-m_1 x} R(x) dx + \frac{e^{m_2 x}}{m_2 - m_1} \int e^{-m_2 x} R(x) dx$
- ii.  $y_p = x e^{mx} \int e^{-mx} R(x) dx - e^{mx} \int x e^{-mx} R(x) dx$
- iii.  $y_p = \frac{e^{px} \sin qx}{q} \int e^{-px} R(x) \cos qx dx - \frac{e^{px} \cos qx}{q} \int e^{-px} R(x) \sin qx dx$

### **Bernoulli's Differential Equation**

$$\frac{dy}{dx} + P(x)y = Q(x)y^n$$

Let

$$v = y^{1-n}$$

$$v = \frac{(1-n) \int Q e^{(1-n) \int P dx} dx + c}{e^{(1-n) \int P dx}}$$

## Euler Differential Equation

$$x^2 \frac{d^2y}{dx^2} + ax \frac{dy}{dx} + by = s(x)$$

Change variable  $x = e^t$ . After some algebra we find

$$\frac{d^2y}{dt^2} + (a-1)\frac{dy}{dt} + by = s(e^t)$$

which is a constant coefficient equation.

An alternative method that is more convenient is to assume a power law solution. That is

$$y = Ax^m$$

which leads to a characteristic equation given as

$$m(m-1) + am + b = 0$$

Solving for  $m = m_1, m_2$ . Then

$$y = A_1x^{m_1} + A_2x^{m_2}$$

## Homogeneous Differential Equations

$$\frac{dy}{dx} = F\left(\frac{y}{x}\right)$$

Set  $v = \frac{y}{x}$ , then

$$\ln x = \int \frac{dv}{F(v) - v} + c$$

### Exact Differential Equations

$$M(x, y)dx + N(x, y)dy = 0$$

with

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} = \frac{\partial^2 \varphi(x, y)}{\partial x \partial y}$$

i.e.

$$M = \frac{\partial \varphi}{\partial x}, \quad N = \frac{\partial \varphi}{\partial y}.$$

The solution then is given by

$$\varphi(x, y) = \text{const}$$

### Riccati Differential Equation

$$\frac{dy}{dx} = A(x)y^2 + B(x)y + C(x)$$

Let

$$y = -\frac{1}{A(x)v} \frac{dv}{dx}$$

Then

$$\frac{d^2v}{dx^2} - \left[ \frac{1}{A} \frac{dA}{dx} + B(x) \right] \frac{dv}{dx} + A(x)C(x)v = 0,$$

which is a linear equation.

If a solution  $y_0$  of the Riccati Equation is known, then the transformation

$$y = y_0(x) + u$$

leads to a Bernoulli's Equation. i.e.,

$$\frac{du}{dt} = (B + 2Ay_0)u + Au^2$$

### Bessel's Differential Equation

$$x^2 \frac{d^2y}{dx^2} + x \frac{dy}{dx} + (\beta^2 x^2 - n^2)y = 0$$

$$y = C_1 J_n(\beta x) + C_2 Y_n(\beta x)$$

where  $J_n(\beta x)$ ,  $Y_n(\beta x)$  are the Bessel functions of the first and second kinds, respectively.

### Generalized Bessel's Equation

$$\frac{d^2y}{dx^2} + \frac{1-2a}{x} \frac{dy}{dx} + \left[ (bcx^{c-1})^2 + \frac{a^2 - m^2 c^2}{x^2} \right] y = 0$$

$$y = x^a (c_1 J_m(bx^c) + c_2 Y_m(bx^c))$$

### Modified Bessel's Differential Equation

$$x^2 \frac{d^2y}{dx^2} + x \frac{dy}{dx} - (\beta^2 x^2 + n^2)y = 0$$

$$y = c_1 I_n(\beta x) + c_2 K_n(\beta x)$$

where  $I_n$  and  $K_n$  are the modified Bessel functions.

### Legendre's Differential Equation

$$\frac{d}{dx} \left[ (1-x^2) \frac{dy}{dx} \right] + n(n+1)y = 0$$

$$y = c_1 P_n(x) + c_2 Q_n(x)$$

where  $P_n$  and  $Q_n$  are the Legendre functions and

$$P_0 = 1, P_1 = x, P_2 = \frac{1}{2}(3x^2 - 1), P_3 = \frac{1}{2}(5x^3 - 3x)$$

### Associated Legendre's Differential Equation

$$\frac{d}{dx} \left[ (1-x^2) \frac{dy}{dx} \right] + [n(n+1) - \frac{m^2}{(1-x^2)}] y = 0$$

$$y = c_1 P_n^m(x) + c_2 Q_n^m(x)$$

where  $P_n^m$  and  $Q_n^m$  are the associated Legendre functions where

$$P_1^1 = (1-x^2)^{1/2}, \quad P_2^1 = 3x(1-x^2)^{1/2}, \quad P_2^2 = 3(1-x^2).$$

Also

$$P_n^0 = P_n.$$

### Hermite's Differential Equation

$$\frac{d^2 y}{dx^2} - 2x \frac{dy}{dx} + 2ny = 0$$

The solutions are the Hermite polynomials,  $H_n(x)$ . These are

$$H_0(x) = 1, \quad H_1(x) = 2x \quad H_2(x) = 4x^2 - 2, \quad H_3(x) = 8x^3 - 12x.$$

### Laguerre's Differential Equation

$$\frac{d^2 y}{dx^2} + (1-x) \frac{dy}{dx} + ny = 0$$

The solutions are the Laguerre polynomials,  $L_n(x)$ . These are

$$L_0(x) = 1, \quad L_1(x) = 1-x \quad L_2(x) = x^2 - 4x + 2.$$

### Associated Laguerre's Differential Equation

$$x \frac{d^2 y}{dx^2} + (m+1-x) \frac{dy}{dx} + (n-m)y = 0.$$

Solutions are the associated Laguerre polynomials  $L_n^m(x)$ :

$$\begin{aligned} L_1^1 &= -1, & L_2^1 &= 2x - 4, & L_2^2 &= 2, & L_3^1 &= -3x^2 + 18x - 18, \\ L_3^2 &= -6x + 18, & L_3^3 &= -6. \end{aligned}$$

### Chebyshev's Differential Equation

$$(1-x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} + n^2y = 0.$$

Solutions are the Chebyshev polynomials given as

$$T_n(x) = \cos(n \cos^{-1} x), \quad T_0 = 1, T_1 = x, T_2 = 2x^2 - 1, T_3 = 4x^2 - 3x$$

$$y = \begin{cases} c_1 T_n(x) + B\sqrt{1-x^2} U_{n-1}(x) & n \neq 0 \\ A + B \sin^{-1} x & n = 0 \end{cases}$$

Here  $U_n(x)$  is the Chebyshev polynomials of the second kind.

### Hypergeometric Differential Equations

$$x(1-x)\frac{d^2y}{dx^2} + [c - (a+b+1)x]\frac{dy}{dx} - aby = 0$$

Solution is the hypergeometric function  $F(a, b; c; x)$ .



## Fourier Series

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left( a_n \cos \frac{n\pi x}{L} + b_n \sin \frac{n\pi x}{L} \right)$$

$$a_n = \frac{1}{L} \int_{-L}^L f(x) \cos \frac{n\pi x}{L} dx, \quad b_n = \frac{1}{L} \int_{-L}^L f(x) \sin \frac{n\pi x}{L} dx$$

### Fourier Cosine Series

When  $f(x) = f(-x)$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{L}, \quad a_n = \frac{2}{L} \int_0^L f(x) \cos \frac{n\pi x}{L} dx$$

### Fourier Sine Series

When  $f(x) = -f(-x)$

$$f(x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{L}, \quad b_n = \frac{2}{L} \int_0^L f(x) \sin \frac{n\pi x}{L} dx$$

### Fourier Exponential Series

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{i\omega_n x}, \quad \text{where} \quad \omega_n = \frac{n\pi}{L},$$

$$c_n = \frac{1}{2L} \int_{-L}^L f(x) e^{-i\omega_n x} dx$$

### Summary of Properties of Fourier Transforms

Consider the Fourier Exponential Series in the region  $-L < x < L$

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{\frac{i n \pi x}{L}} \quad -L < x < L,$$

$$c_n = \frac{1}{2L} \int_{-L}^L e^{-\frac{i n \pi x}{L}} f(x) dx$$

Replacing the expression for the coefficient  $c_n$  in the series, we find

$$f(x) = \frac{1}{2L} \sum_{-\infty}^{+\infty} \int_{-L}^L e^{i\omega_n(x-x')} f(x') dx' .$$

In the last step we have defined

$$\omega_n = \frac{n\pi}{L}, \quad \Delta\omega = \frac{\pi}{L} .$$

As  $L \rightarrow \infty$ ,  $\sum g_n \Delta\omega = \int g d\omega$ . Thus,

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} e^{i\omega(x-x')} f(x') dx' d\omega \quad \text{Fourier Integral Representation}$$

Define Fourier Transform (Exponential)

$$\bar{f}(\omega) = \int_{-\infty}^{+\infty} e^{-i\omega x'} f(x') dx'$$

The inverse transform is

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{i\omega x} \bar{f}(\omega) d\omega$$

The above two equations are a Fourier Exponential Transform Pair.

Fourier Integral Representation (FIR) may be restated as

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \cos \omega(x-x') f(x') dx' d\omega ,$$

or

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} (\cos \omega x \cos \omega x' + \sin \omega x \sin \omega x') f(x') dx' d\omega .$$

For even functions (i.e.  $f(x) = f(-x)$ ), and FIR becomes

$$f(x) = \frac{2}{\pi} \int_0^{\infty} \int_0^{\infty} \cos \omega x \cos \omega x' f(x') dx' d\omega$$

*Definition:* Fourier-Cos Transform Pair:

$$\bar{f}_c(\omega) = \int_0^{\infty} \cos \omega x' f(x') dx'$$

$$f(x) = \frac{2}{\pi} \int_0^{\infty} \cos \omega x \bar{f}_c(\omega) d\omega$$

For odd functions (i.e.  $f(x) = -f(-x)$ ), and FIR becomes

$$f(x) = \frac{2}{\pi} \int_0^{\infty} \int_0^{\infty} \sin \omega x \sin \omega x' f(x') dx'$$

*Definition:* Fourier-Sin Transform Pair:

$$\bar{f}_s(\omega) = \int_0^{\infty} \sin \omega x' f(x') dx'$$

$$f(x) = \frac{2}{\pi} \int_0^{\infty} \sin \omega x \bar{f}_s(\omega) d\omega$$

### Applications to Differential Equations

Fourier Exponential Transform of derivatives

$$\mathfrak{F}\left\{\frac{df}{dx}\right\} = \int_{-\infty}^{+\infty} e^{-i\omega x} \frac{df(x)}{dx} dx = i\omega \bar{f}(\omega)$$

$$\mathfrak{F}\left\{\frac{d^2f}{dx^2}\right\} = -\omega^2 \bar{f}(\omega), \quad \mathfrak{F}\left\{\frac{d^nf}{dx^n}\right\} = (i\omega)^n \bar{f}(\omega)$$

*Example:* Find  $f$  that satisfies the following differential equation:

$$\frac{d^2f}{dx^2} + a \frac{df}{dx} + bf = \delta(x - x_0) \quad -\infty < x < +\infty$$

Take Fourier Exponential Transform

$$-\omega^2 \bar{f}(\omega) + ai\omega \bar{f}(\omega) + b\bar{f}(\omega) = e^{-i\omega x_0}$$

$$\bar{f}(\omega) = \frac{e^{-i\omega x_0}}{b - \omega^2 + ia\omega}$$

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \frac{e^{i\omega(x-x_0)}}{b - \omega^2 + ia\omega} d\omega$$

## Laplace Transform

Consider the class of functions, which are zero for negative  $x$ , and are defined according to

$$f(x) = \begin{cases} F(x)e^{-\gamma x} & x \geq 0 \\ 0 & x < 0 \end{cases} \quad \gamma > 0$$

FIR of  $f(x)$  for  $x \geq 0$  becomes

$$F(x)e^{-\gamma x} = \frac{1}{2\pi} \int_{-\infty}^{+\infty} d\omega \int_0^{\infty} e^{i\omega(x-x')} F(x') e^{-\gamma x'} dx'$$

or

$$F(x) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} d\omega e^{(\gamma+i\omega)x} \int_0^{\infty} e^{-(\gamma+i\omega)x'} F(x') dx'$$

Let

$$\gamma + i\omega = S, \quad d\omega = \frac{ds}{i}$$

$$F(x) = \frac{1}{2\pi i} \int_{\gamma-i\omega}^{\gamma+i\omega} ds e^{sx} \int_0^{\infty} e^{-sx'} F(x') dx'$$

*Definition:* Laplace Transform

$$\bar{F}(s) = \int_0^{\infty} e^{-sx'} F(x') dx'$$

Inverse Transform

$$F(x) = \frac{1}{2\pi i} \int_{\gamma-i\omega}^{\gamma+i\omega} e^{sx} \bar{F}(s) ds$$

**Table of Fourier Exponential Transform Pair**

$$\bar{f}(\omega) = \int_{-\infty}^{+\infty} e^{-i\omega x} f(x) dx, \quad f(x) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{i\omega x} \bar{f}(\omega) d\omega$$

$f(x)$	$\bar{f}(\omega)$
$f_1(x)e^{-i\omega_0 x}$	$\bar{f}_1(\omega + \omega_0)$
$f_1(x + x_0)$	$e^{i\omega x_0} \bar{f}(\omega)$
$f_1(x) * f_2(x) = \int_{-\infty}^{+\infty} f_1(\xi) f_2(x - \xi) d\xi$	$\bar{f}_1(\omega) \bar{f}_2(\omega)$
$\delta(x - x_0)$	$e^{-i\omega x_0}$
$e^{i\omega_0 x}$	$2\pi \delta(\omega - \omega_0)$
$e^{-\alpha x }$	$\frac{2\alpha}{\omega^2 + \alpha^2}$
$\cos \omega_0 x$	$\pi[\delta(\omega - \omega_0) + \delta(\omega + \omega_0)]$
$e^{-\alpha x } \cos \beta x$	$\frac{2\alpha(\omega^2 + \alpha^2 + \beta^2)}{(\omega^2 - \beta^2 - \alpha^2)^2 + 4\alpha^2 \omega^2}$
$e^{-\alpha x } \left[ \cos \beta x + \frac{\alpha}{\beta} \sin \beta x  \right]$	$\frac{4\alpha(\alpha^2 + \beta^2)}{(\omega^2 - \beta^2 - \alpha^2)^2 + 4\alpha^2 \omega^2}$
$e^{-\alpha^2 x^2} \cos \beta x$	$\frac{\sqrt{\pi}}{2\alpha} \left[ \exp\left\{-\frac{(\omega + \beta)^2}{4\alpha^2}\right\} + \exp\left\{-\frac{(\omega - \beta)^2}{4\alpha^2}\right\} \right]$
$e^{-\alpha^2 x^2}$	$\frac{\sqrt{\pi}}{\alpha} \exp\left\{-\frac{\omega^2}{4\alpha^2}\right\}$
$\frac{d^n}{dx^n} \delta(x)$	$(i\omega)^n$
$J_0(x)$	$\left\{ \begin{array}{ll} \frac{2}{\sqrt{1-\omega^2}} &  \omega  < 1 \\ 0 & \text{elsewhere} \end{array} \right\}$

Table of Laplace Transform Pair

$F(x)$	$\bar{F}(s)$
$e^{ax} F(x)$	$\bar{F}(s-a)$
$F(x-a)U(x-a)$	$e^{-as}\bar{F}(s)$
$\frac{dF}{dx}$	$s\bar{F}(s)-F(0)$
$\frac{d^n F}{dx^n}$	$s^n \bar{F}(s) - s^{n-1}F(0) \dots - F(0)^{(n-1)}$
$x F(x)$	$\frac{d\bar{F}(s)}{ds}$
$x^n F(x)$	$(-1)^n \frac{d^n \bar{F}(s)}{ds^n}$
$\int_0^x F(t)dt$	$\frac{\bar{F}(s)}{s}$
$F_1(x)*F_2(x) = \int_0^x F_1(x-t)F_2(t)dt$	$\bar{F}_1(s)\bar{F}_2(s)$
$\frac{F(x)}{x}$	$\int_s^\infty \bar{F}(s_1)ds_1$
$1, x, x^n$	$\frac{1}{s}, \frac{1}{s^2}, \frac{n!}{s^n}$
$e^{ax}, \sin ax, \cos ax$	$\frac{1}{s-a}, \frac{a}{s^2+a^2}, \frac{s}{s^2+a^2}$
$\sinh ax, \cosh ax$	$-\frac{a}{s^2-a^2}, \frac{s}{s^2-a^2}$
$x \sin ax, x \cos ax$	$\frac{2as}{(s^2+a^2)^2}, \frac{s^2-a^2}{(s^2+a^2)^2}$
$J_0(ax), I_0(ax)$	$\frac{1}{\sqrt{s^2+a^2}}, \frac{1}{\sqrt{s^2-a^2}}$
$\frac{\cos 2\sqrt{ax}}{\sqrt{\pi x}}, \frac{\sin 2\sqrt{ax}}{\sqrt{\pi x}}$	$\frac{e^{-a/s}}{\sqrt{s}}, \frac{e^{-a/s}}{s^{3/2}}$
$\delta(x-a), U(x-a)$	$e^{-as}, \frac{e^{-as}}{s}$
$\frac{1}{\sqrt{\pi x}} e^{-\frac{a^2}{4x}}$	$\frac{e^{-a\sqrt{s}}}{\sqrt{s}}$

## Probability and Random Processes

In this section, capital letters identifies a random variable and lower case letters are used for coordinate systems.

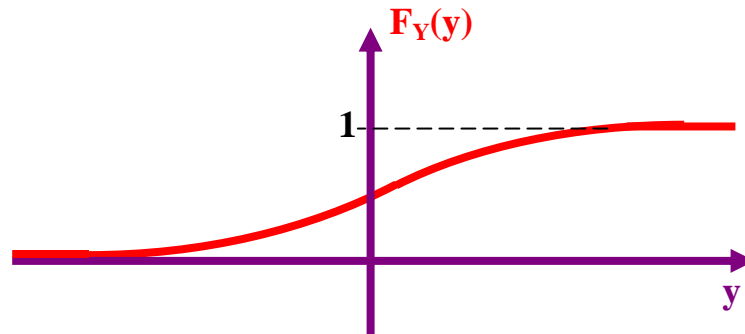
### Distribution Function

The distribution function of a random variable  $Y$  is defined as the probability that  $\{Y \leq y\}$ . That is

$$F_Y(y) = P\{Y \leq y\}$$

It then follows that  $F_Y(y)$  is monotonically increasing function and

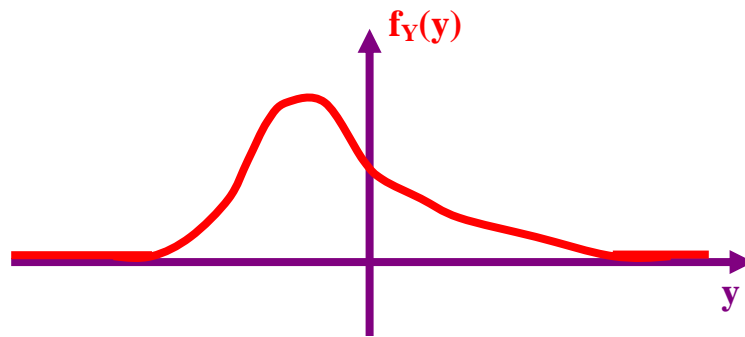
$$0 \leq F_Y(y) \leq 1.$$



### Density Function

The probability density function is defined as

$$f_Y(y) = \frac{dF_Y(y)}{dy}$$



*Properties:*

$$F_Y(\infty) = 1 = \int_{-\infty}^{+\infty} f_Y(y) dy, \quad P\{y_1 < Y \leq y_2\} = \int_{y_1}^{y_2} f_Y(y) dy = F_Y(y_2) - F_Y(y_1)$$

### Expected Value

$$E\{Y\} = \bar{Y} = \int_{-\infty}^{+\infty} y f_Y(y) dy$$

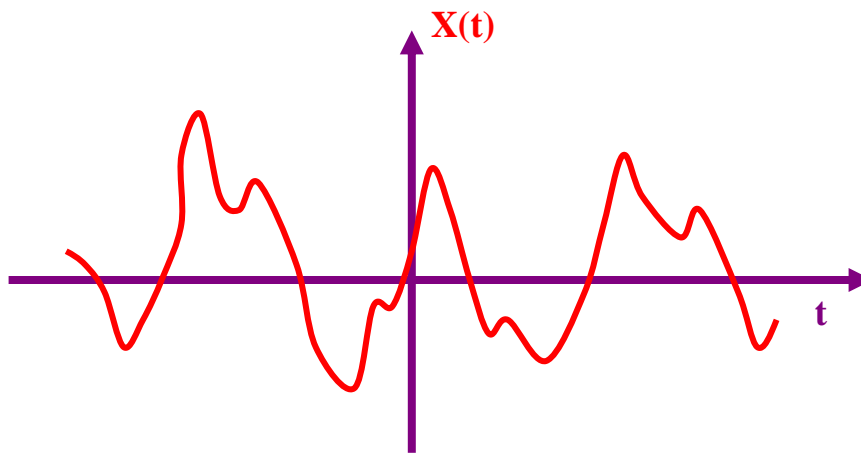
$$E\{g(Y)\} = \overline{g(Y)} = \int_{-\infty}^{+\infty} g(y) f_Y(y) dy$$

## Variance

$$\sigma_Y^2 = E\{(Y - \bar{Y})^2\} = E\{Y^2\} - \bar{Y}^2$$

## Stochastic Process

Ensembles of random functions of time (or space) are referred to as stochastic processes. For fixed time, a stochastic process becomes a random variable. Every sample of a stochastic process is a time function.



Statistics of a stochastic process may be evaluated similar to those of a random variable. For example, the mean value is given as

$$E\{X(t)\} = \int_{-\infty}^{+\infty} xf_x(x, t)dx$$

## Time Average

Time averaging over an interval (0,T) is defined as

$$\bar{X}(t) = \frac{1}{T} \int_0^T X(t)dt \approx E\{X(t)\}$$

## Autocorrelation

The autocorrelation of a random process is defined as

$$R_{xx}(\tau) = E\{X(t + \tau)X(t)\} = \frac{1}{T} \int_0^T X(t + \tau)X(t)dt$$



where  $\tau$  is the time difference, and it is assumed that  $X(t)$  is a stationary random process. Note that

$$R_{xx}(0) = E\{X^2(t)\} = \overline{X^2(t)}$$

### Energy Spectrum

$$S_{xx}(\omega) = \int_{-\infty}^{+\infty} e^{-i\omega\tau} R_{xx}(\tau) d\tau$$

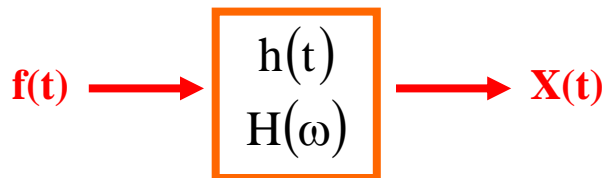
$$R_{xx}(\tau) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{i\omega\tau} S_{xx}(\omega) d\omega$$

It may be shown that

$$S_{xx}(\omega) = \frac{1}{T} |\tilde{X}(\omega)|^2, \quad \tilde{X}(\omega) = \int_{-\infty}^{+\infty} e^{-i\omega t} X(t) dt$$

### Linear Systems

Consider a linear system with impulse response  $h(t)$  and a system function  $H(\omega)$  for  $X(0) = 0$  as shown schematically in the figure.



The solution then is given as

$$X(t) = \int_0^t h(t - \tau) f(\tau) d\tau$$

where

More generally,

$$X(t) = \int_{-\infty}^{+\infty} h(t - \tau) f(\tau) d\tau = h(t) * f(t)$$

Taking Fourier Transform

$$H(\omega) = \int_{-\infty}^{+\infty} e^{-i\omega t} h(t) dt$$

$$\tilde{x}(\omega) = H(\omega)\tilde{f}(\omega)$$

$$S_{xx}(\omega) = \frac{1}{T} |\tilde{x}(\omega)|^2 = \frac{1}{T} |H(\omega)|^2 |\tilde{f}(\omega)|^2 = |H(\omega)|^2 S_{ff}(\omega)$$

$$S_{xx}(\omega) = |H(\omega)|^2 S_{ff}(\omega)$$

For

$$\dot{x} + \alpha x = f(t),$$

$$h(t) = e^{-\alpha t}$$

For

$$\ddot{x} + 2\zeta\omega_0\dot{x} + \omega_0^2 x = f(t),$$

$$h(t) = \frac{1}{\omega_d} e^{-\zeta\omega_0 t} \sin \omega_d t \quad \omega_d = \omega_0 \sqrt{1 - \zeta^2}$$

## Useful Integrals

$$\int e^{ax} P(x) dx = \frac{e^{ax}}{a} \left[ P(x) - \frac{1}{a} P'(x) + \frac{1}{a^2} P''(x) - \frac{1}{a^3} P'''(x) \dots \right]$$

$$\int e^{ax} \sin bx dx = \frac{e^{ax} (a \sin bx - b \cos bx)}{a^2 + b^2}$$

$$\int e^{ax} \cos bx dx = \frac{e^{ax} (a \cos bx + b \sin bx)}{a^2 + b^2}$$

$$\int x \sin ax dx = \frac{\sin ax - ax \cos ax}{a^2}$$

$$\int x \cos ax dx = \frac{\cos ax + ax \sin ax}{a^2}$$

$$\int \ln ax dx = x \ln ax - x$$

$$\int x \ln ax dx = \frac{x^2}{2} \ln ax - \frac{x^2}{4}$$

$$\int x \sinh ax dx = \frac{ax \cosh ax - \sinh ax}{a^2}$$

$$\int x \cosh ax dx = \frac{ax \sinh ax - \cosh ax}{a^2}$$

$$\int \frac{dx}{a^2 + x^2} = \frac{1}{a} \tan^{-1} \left( \frac{x}{a} \right), \quad \int \frac{dx}{a^2 - x^2} = \frac{1}{a} \tanh^{-1} \left( \frac{x}{a} \right) = \frac{1}{2a} \ln \frac{x-a}{x+a}$$

$$\int \frac{dx}{\sqrt{a^2 - x^2}} = \sin^{-1} \left( \frac{x}{a} \right)$$

$$\int \frac{dx}{\sqrt{x^2 \pm a^2}} = \ln \left( x + \sqrt{x^2 \pm a^2} \right)$$

$$\int e^{ax} x dx = \frac{e^{ax}}{a} \left( x - \frac{1}{a} \right)$$

### Vector Identities

$$\nabla \cdot \nabla \times \vec{u} = 0 \quad \nabla \times (\nabla \phi) = 0$$

$$\nabla \times \nabla \times \vec{u} = \nabla(\nabla \cdot \vec{u}) - \nabla^2 \vec{u}$$

$$\vec{u} \cdot \nabla \vec{u} = \nabla \left( \frac{u^2}{2} \right) - \vec{u} \times (\nabla \times \vec{u})$$

$$\nabla \times (\vec{u} \times \vec{v}) = \vec{v} \cdot \nabla \vec{u} - \vec{u} \cdot \nabla \vec{v} + (\nabla \cdot \vec{v})\vec{u} - (\nabla \cdot \vec{u})\vec{v}$$

$$\nabla \cdot (\vec{u} \times \vec{v}) = \vec{v} \cdot \nabla \times \vec{u} - \vec{u} \cdot \nabla \times \vec{v}$$

$$\nabla(\vec{u} \cdot \vec{v}) = \vec{v} \cdot \nabla \vec{u} + \vec{u} \cdot \nabla \vec{v} + \vec{v} \times (\nabla \times \vec{u}) + \vec{u} \times (\nabla \times \vec{v})$$

### Stokes Theorem

$$\oint_C \vec{u} \cdot d\vec{c} = \int_S (\nabla \times \vec{u}) \cdot d\vec{s}$$

### Divergence Theorem

$$\int_V \nabla \cdot \vec{u} dV = \int_S \vec{u} \cdot d\vec{s}$$