ME 529 - Stochastics

Jointly Normal Random Variables

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Gaussian Random Variables

Outline

- Jointly Normal Density Function
- Covariance Matrix
- Important Properties
- Schwartz Inequality
- Holder Inequality

Gaussian Random Variables

Random variables $X_1, X_{12}, \ldots, X_n$ are jointly normal (Gaussian) if their joint density is given as

$$f_{\mathbf{X}}(\mathbf{x}) = (2\pi)^{-\frac{n}{2}} |\Lambda|^{-\frac{1}{2}} \exp\left\{-\frac{1}{2}(\mathbf{x}-\boldsymbol{\eta})^T \cdot \boldsymbol{\Lambda}^{-1} \cdot (\mathbf{x}-\boldsymbol{\eta})\right\}$$

$$\eta = E\{\mathbf{x}\}$$

n × n covariance matrix

$$\boldsymbol{\Lambda} = [\mu_{ij}] \qquad \mu_{ij} = E\{(X_i - \eta_i)(X_j - \eta_j)\} \qquad |\Lambda| = \det|\Lambda|$$

Gaussian Random Variables

Jointly normal density function rewritten as:

$$f_{\mathbf{X}}(\mathbf{x}) = \frac{1}{(2\pi)^{\frac{n}{2}} |\Lambda|^{\frac{1}{2}}} e^{-\frac{1}{2}\sum_i \sum_j \Lambda_{ij}^{-1}(x_i-\eta_i)(x_j-\eta_j)}$$

Corresponding Characteristic Function:

$$\Phi_{\mathbf{X}}(\boldsymbol{\omega}) = e^{i\boldsymbol{\eta}^T \cdot \boldsymbol{\omega} - \frac{1}{2}\boldsymbol{\omega}^T \cdot \boldsymbol{\Lambda} \cdot \boldsymbol{\omega}}$$

$$\Phi_X(\boldsymbol{\omega}) = exp\left\{i\sum_j \eta_j \omega_j - \frac{1}{2}\sum \Lambda_{k\ell}\omega_k\omega_\ell\right\}$$

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Important Properties

1. η, Λ given ⇨ $f_X(x)$ fully specified
2. If $E\{X\} = 0$ $E\{X_1^{k_1} X_2^{k_2} \dots X_n^{k_n}\} = 0$ if $\sum_j k_j$ ~ is odd
3. Even moments: $E\{X_1 X_2 \dots X_n\} = \sum_{m_1,\dots,m_n} E\{X_{m1} X_{m2}\} \dots E\{X_{m_{n-1}} X_{m_n}\}$

   Note: sum taken over all combinations of n/2 pairs of *n* random variables as follows

   $$E\{X_i X_j X_k X_m\} = E\{X_i X_j\}E\{X_k X_m\} + E\{X_i X_k\}E\{X_j X_m\} + E\{X_i X_m\}E\{X_j X_k\}$$
4. If X_i are normal so are $Y_j = \sum_{i=1}^{n} c_{ji} X_i$ j = 1, 2, …

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Note: Linear transformation of normal random variables leads to a set of new normal random variables.

Schwarz Inequality

$$E\{|XY|\} \le \left[E\{X^2\}E\{Y^2\}\right]^{\frac{1}{2}}$$

Holder Inequality

$$E\{|XY|\} \le \left[E\{|X|^n\}\right]^{\frac{1}{n}} \left[E\{|Y|^m\}\right]^{\frac{1}{m}}$$

$n, m > 0$ $\frac{1}{n} + \frac{1}{m} = 1$

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Concluding Remarks

- Jointly Normal Density Function
- Covariance Matrix
- Important Properties
- Schwartz Inequality
- Holder Inequality

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Thank you!

Questions?