ME 529 - Stochastics

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Central Limit Theorem

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Central Limit Theorem

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Outline

- Central Limit Theorem
- Proof

Central Limit Theorem

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Let $X_1,\ldots,X_n$ be a sequence of mutually independent and identically distributed random variables with means η and variances σ^2.

Let

$$X = \sum_{j=1}^{n} X_j$$

Define normalized random variable

$$Y = \frac{X - n\eta}{\sigma n^{1/2}}$$

As $n \to \infty$ distribution function of Y converges to zero mean unit variance Gaussian distribution.

Central Limit Theorem-Proof

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$$\Phi_Y(\omega) = E\left\{e^{i\omega Y}\right\} = E\left\{e^{i\Omega(X-n\eta)}\right\} = E\left\{e^{i\Omega\sum_{j=1}^{n}(X_j-\eta)}\right\} = E\left\{\prod_{j=1}^{n} e^{i\Omega(X_j-\eta)}\right\}$$

$$\Omega = \frac{\omega}{\sigma n^{1/2}}$$

Noting that $X_1,\ldots,X_n$ are independent

$$\Phi_Y(\omega) = \prod_{j=1}^{n} E\left\{e^{i\Omega(X_j-\eta)}\right\} = \prod_{j=1}^{n}\left(e^{-i\Omega\eta} E\left\{e^{i\Omega X_j}\right\}\right)$$
$$= \prod_{j=1}^{n}\left(e^{-i\Omega\eta}\Phi_{Xj}(\Omega)\right) = \left(e^{-i\Omega\eta}\Phi_{Xj}(\Omega)\right)^n$$

Also note

$$\Phi_{Xj}(\Omega) = \Phi_{X1}(\Omega) = \ldots = \Phi(\Omega)$$

Central Limit Theorem-Proof

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As $n \to \infty, \Omega \to 0,$

$$\Phi(\Omega) = \sum_k m_k \frac{(i\Omega)^k}{k!} \qquad m_k = E\{X^k\}$$

$$\Phi_Y(\omega) = \left[\left(1 - i\Omega\eta - \frac{\Omega^2\eta^2}{2}\ldots\right)\left(1 + i\Omega\eta - \frac{\sigma^2 + \eta^2}{2}\Omega^2\ldots\right)\right]^n$$

$$= \left[1 + \Omega^2\eta^2 - \frac{\Omega^2\eta^2}{2} - \frac{\sigma^2 + \eta^2}{2}\Omega^2\ldots\right]^n = \left(1 - \frac{\sigma^2\eta^2}{2}\ldots\right)^n$$

Noting $\lim_{N\to\infty}\left(1 + \frac{c}{N}\right)^N \to e^c$ $\Rightarrow$ $\Phi_Y(\omega) = \left(1 - \frac{\omega^2}{2n}\right)^n \to e^{-\omega^2/2}$

Central Limit Theorem

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Thank you!

Questions?