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Transformations of Random Variables

Transformations of Two Random Variables

Given the joint density of random variables X and Y, $f_{XY}(x, y)$, and the functional relationships $Z = g(X, Y)$, $W = h(X, Y)$, we want to find $f_{ZW}(z, w)$.

Theorem 1: To find $f_{ZW}(z, w)$, solve equations

$$\begin{aligned} g(x, y) &= z \\ h(x, y) &= w \end{aligned},$$

for x and y in terms of z and w. If $(x_1, y_1), \ldots, (x_n, y_n), \ldots$ are real solutions of these equations, that is, $g(x_i, y_i) = z$, $h(x_i, y_i) = w$ then $f_{ZW}(z, w)$ is given by

$$f_{ZW}(z, w) = \sum_i \frac{f_{XY}(x_i, y_i)}{|J(x_i, y_i)|},$$

where

$$J(x, y) = \begin{vmatrix} \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \\ \frac{\partial h}{\partial x} & \frac{\partial h}{\partial y} \end{vmatrix},$$

is the Jacobian of transformation. If for certain values of (z, w) there is no real solution, then $f_{ZW}(z, w) = 0$. (For proof see Papoulis, pp. 201-202)

Auxiliary Variables

To find the density of a function of two random variables, $Z = g(X, Y)$, introduce an auxiliary variable $W = X$ or $W = Y$. Find the joint density of Z and W by the use of Theorem 1. Then

$$f_Z(z) = \int_{-\infty}^{+\infty} f_{ZW}(z, w)\, dw$$

Transformations of Several Random Variables

Given the joint density, $f(x_1, \ldots, x_n)$ and $Y_1 = g_1(x_1, \ldots, x_n), \ldots, Y_k = g_k(x_1, \ldots, x_n)$, we want to find the joint density of $f(y_1, \ldots, y_n)$.

Theorem 2

To find $f_{\underline{\mathbf{Y}}}(\underline{\mathbf{y}})$, if $k < n$, first introduce auxiliary variables

$$Y_{k+1} = X_{k+1}, \ldots Y_n = X_n,$$

which increases the number of **Y**s to n. Then solve equations

$$g_i(\underline{\mathbf{x}}) = y_i, \qquad i = 1, \ldots, n.$$

If $\underline{x}_j$ ($j = 1,2,\ldots$) are real solutions, then

$$f_{\underline{Y}}(\underline{\mathbf{y}}) = \sum_{\substack{j \\ Real\ Solutions}} \frac{f_{\underline{X}}(\mathbf{x}_j)}{\left| J(\underline{\mathbf{x}}_j) \right|},$$

where

$$J(\underline{\mathbf{x}}) = \begin{vmatrix} \frac{\partial g_1}{\partial x_1} & \ldots & \frac{\partial g_1}{\partial x_n} \\ \ldots & & \ldots \\ \frac{\partial g_n}{\partial x_1} & \ldots & \frac{\partial g_n}{\partial x_n} \end{vmatrix} = Jacobian.$$

If there is no real solution (for certain values of $\underline{\mathbf{y}}$), then

$$f_{\underline{Y}}(\underline{\mathbf{y}}) = 0.$$

Method of Characteristic Function

To find the density of $Z = g(x_1, \ldots, x_n)$, one option is to find the characteristic function of Z first. i.e.,

$$\Phi_Z(\omega) = E\{e^{i\omega Z}\} = E\{e^{i\omega g(\underline{X})}\} = \int_{-\infty}^{+\infty} \ldots \int_{-\infty}^{+\infty} e^{ig(\underline{x})} f_{\underline{X}}(\underline{x}) dx_1 \ldots dx_n.$$

Then

$$f_Z(z) = \Im^{-1}\{\Phi_Z(\omega)\} = \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{-i\omega Z} \Phi_Z(\omega) d\omega.$$