

# Transformations of Two Random Variables

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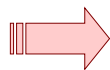
## Outline

- Transformations of Two Random Variables
- Auxiliary Variables
- Transformations of Several Random Variables
- Method of Characteristic Function

Given  $f_{XY}(x,y)$  and  $Z = g(X,Y)$ ,  $W = h(X,Y)$ , find  $f_{ZW}(z,w)$ .

Solve

$$\begin{cases} g(x,y) = z \\ h(x,y) = w \end{cases}$$



Find real solutions  $(x_i, y_i)$

Then

$$f_{ZW}(z,w) = \sum_i \frac{f_{XY}(x_i, y_i)}{|J(x_i, y_i)|}$$

Where

$$J(x,y) = \begin{vmatrix} \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \\ \frac{\partial h}{\partial x} & \frac{\partial h}{\partial y} \end{vmatrix}$$

If for certain values of  $(z,w)$  there is no real solution,  $f_{ZW}(z,w) = 0$

To find the density of a function of two random variables,  $Z = g(X,Y)$ , introduce an auxiliary variable  $W = X$  or  $W = Y$ . Find the joint density of  $Z$  and  $W$  by the use of Theorem 1. Then

$$f_Z(z) = \int_{-\infty}^{+\infty} f_{ZW}(z,w) dw$$

# Several Random Variables Clarkson University

Given  $f_{\mathbf{X}}(x_1, \dots, x_n)$  and  $Y_1 = g_1(x_1, \dots, x_n)$ ,  
 $\dots, Y_k = g_k(x_1, \dots, x_n)$ , find  $f_{\mathbf{Y}}(\mathbf{y})$ ,

To find  $f_{\mathbf{Y}}(\mathbf{y})$ , if  $k < n$ , introduce auxiliary variables  
 $Y_{k+1} = X_{k+1}, \dots, Y_n = X_n$

Now solve

$$g_i(\mathbf{x}) = y_i \quad i = 1, \dots, n$$

# Theorem 2 Clarkson University

If  $x_j$  ( $j = 1, 2, \dots$ ) are real solutions, then

$$f_{\mathbf{Y}}(\mathbf{y}) = \sum_j \frac{f_{\mathbf{X}}(\mathbf{x}_j)}{|J(\mathbf{x}_j)|}$$

*Real Solutions*

$$J(\mathbf{x}) = \begin{vmatrix} \frac{\partial g_1}{\partial x_1} & \dots & \frac{\partial g_1}{\partial x_n} \\ \dots & \dots & \dots \\ \frac{\partial g_n}{\partial x_1} & \dots & \frac{\partial g_n}{\partial x_n} \end{vmatrix} = \text{Jacobian}$$

If there is no real solution for certain values of  $\mathbf{y}$ , then

$$f_{\mathbf{Y}}(\mathbf{y}) = 0$$

# Method of Characteristics Clarkson University

To find the probability density of  $Z = g(x_1, \dots, x_n)$   
 one option is to first find the characteristic function of Z

$$\Phi_Z(\omega) = E\{e^{i\omega Z}\} = E\{e^{i\omega g(\mathbf{X})}\} = \int_{-\infty}^{+\infty} \dots \int_{-\infty}^{+\infty} e^{i\omega g(\mathbf{x})} f_{\mathbf{X}}(\mathbf{x}) dx_1 \dots dx_n$$

Then

$$f_Z(z) = \mathfrak{F}^{-1}\{\Phi_Z(\omega)\} = \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{-i\omega z} \Phi_Z(\omega) d\omega$$

# Transformations Clarkson University

## Concluding Remarks

- Transformations of Two Random Variables
- Auxiliary Variables
- Transformations of Several Random Variables
- Method of Characteristic Function