ME 529 - Stochastics

Transformations of Two Random Variables

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Transformations

Outline

- Transformations of Two Random Variables
- Auxiliary Variables
- Transformations of Several Random Variables
- Method of Characteristic Function

Theorem 1

Given $f_{XY}(x,y)$ and $Z = g(X,Y)$, $W = h(X,Y)$, find $f_{ZW}(z,w)$.

Solve

$$g(x,y) = z$$
$$h(x,y) = w$$

Find real solutions (x_i, y_i)

Then

$$f_{ZW}(z,w) = \sum_i \frac{f_{XY}(x_i,y_i)}{|J(x_i,y_i)|}$$

Where

$$J(x,y) = \begin{vmatrix} \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \\ \frac{\partial h}{\partial x} & \frac{\partial h}{\partial y} \end{vmatrix}$$

If for certain values of (z,w) there is no real solution, $f_{ZW}(z,w) = 0$

Auxiliary Variables

To find the density of a function of two random variables, $Z = g(X,Y)$, introduce an auxiliary variable $W = X$ or $W = Y$. Find the joint density of Z and W by the use of Theorem 1. Then

$$f_Z(z) = \int_{-\infty}^{+\infty} f_{ZW}(z,w)\,dw$$

Several Random Variables

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Given $f_{\underline{X}}(x_1,\ldots,x_n)$ and $Y_1 = g_1(x_1,\ldots,x_n)$, $\ldots$, $Y_k = g_k(x_1,\ldots,x_n)$, find $f_{\underline{Y}}(\underline{y})$,

To find $f_{\underline{Y}}(\underline{y})$, if $k < n$, introduce auxiliary variables $Y_{k+1} = X_{k+1}, \ldots Y_n = X_n$

Now solve $g_i(\underline{\mathbf{x}}) = y_i$ $\quad i = 1,\ldots,n$

Theorem 2

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If x_j ($j = 1,2,\ldots$) are real solutions, then

$$f_{\underline{Y}}(\underline{\mathbf{y}}) = \sum_j \frac{f_{\underline{X}}(\mathbf{x}_j)}{|J(\underline{\mathbf{x}}_j)|}$$

Real Solutions

$$J(\underline{\mathbf{x}}) = \begin{vmatrix} \frac{\partial g_1}{\partial x_1} & \ldots & \frac{\partial g_1}{\partial x_n} \\ \ldots & & \ldots \\ \frac{\partial g_n}{\partial x_1} & \ldots & \frac{\partial g_n}{\partial x_n} \end{vmatrix} = Jacobian$$

If there is no real solution for certain values of $\underline{y}$, then

$$f_{\underline{Y}}(\underline{\mathbf{y}}) = 0$$

Method of Characteristics

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To find the probability density of $Z = g(x_1,\ldots,x_n)$ one option is to first find the characteristic function of Z

$$\Phi_Z(\omega) = E\{e^{i\omega Z}\} = E\{e^{i\omega g(\underline{\mathbf{x}})}\} = \int_{-\infty}^{+\infty}\ldots\int_{-\infty}^{+\infty} e^{ig(\underline{\mathbf{x}})} f_{\underline{X}}(\underline{\mathbf{x}})dx_1\ldots dx_n$$

Then

$$f_Z(z) = \Im^{-1}\{\Phi_Z(\omega)\} = \frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{-i\omega z}\Phi_Z(\omega)d\omega$$

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Concluding Remarks

- Transformations of Two Random Variables
- Auxiliary Variables
- Transformations of Several Random Variables
- Method of Characteristic Function