

Characteristic Function

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Outline

- Characteristic Function
- Inversion Formula
- Moment Theorem
- Moment Generating Function

Characteristic Function of Random Variable X

$$\Phi_X(\omega) = E\{e^{i\omega X}\} = \int_{-\infty}^{+\infty} e^{i\omega x} f_X(x) dx$$

For discrete random variable with

$$f(x) = \sum_j P_j \delta(x - x_j)$$



$$\Phi(x) = \sum_j P_j e^{i\omega x_j}$$

Second Characteristic Function of X

$$\psi(\omega) = \ln \Phi(\omega)$$

or

$$\Phi(\omega) = e^{\psi(\omega)}$$

Important Properties

1. $\Phi(\theta) = \int_{-\infty}^{+\infty} f(x) dx = 1$ $\psi(0) = 0$
2. $|\Phi(\omega)| \leq 1$
3. $\Phi(\omega)$ is a positive definite function, i.e. $\sum_{m=1}^n \sum_{k=1}^n \Phi(\omega_m - \omega_k) a_m a_k^* \geq 0$ for any set of complex coefficients a_m .

Characteristic Function

Inversion Formula

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \Phi(\omega) e^{-i\omega x} d\omega$$

If $f(x)$ is an even function $\Phi(\omega)$ is real and even:

$$\Phi(\omega) = \int_{-\infty}^{+\infty} f(x) \cos \omega x dx = \frac{1}{\pi} \int_0^{\infty} \Phi(\omega) \cos \omega x dx$$

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Moment Theorem

Evaluation of moments from $\Phi(\omega)$:

$$\frac{d^n \Phi(0)}{d\omega^n} = i^n E\{x^n\} = i^n m_n$$

or

$$m_n = E\{x^n\} = \frac{1}{i^n} \frac{d^n \Phi(0)}{d\omega^n}$$

Taylor Series Expansion



$$\Phi(\omega) = \sum_{j=0}^{\infty} \frac{m_j}{j!} (i\omega)^j$$

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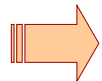
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Moment Generating Function

$$\Phi_X^*(s) = E\{e^{sx}\} = \int_{-\infty}^{+\infty} e^{sx} f_X(x) dx$$

For discrete random variable with

$$f(x) = \sum_j P_j \delta(x - x_j)$$



$$\Phi^*(s) = \sum_i P_i e^{sx_i}$$

Note:

$$\Phi^*(i\omega) = \Phi(\omega)$$

$$\Phi\left(\frac{s}{i}\right) = \Phi^*(s)$$

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Moment Generating Function

Moment Theorem then gives

$$E\{x^n\} = \Phi^{*(n)}(0)$$

$$\Phi^*(s) = \sum_{j=0}^{\infty} \frac{m_j}{j!} s^j$$

If $f(x)$ is zero for $x < 0$:

$$\Phi(s) = \int_0^{\infty} f(x) e^{sx} dx = L\{f(x)\} \Big|_{s=-s}$$

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