Clarkson University

Conditional Distributions and Densities

Definition: The conditional distribution of $X(\xi)$ given (event) m is defined as

$$F_X(x \mid m) = P\{X(\xi) \le x \mid m\} = \frac{P\{(X \le x) \cap m\}}{P\{m\}}.$$

Note that $((X(\xi) \le x) \cap m)$ is the event consisting of all outcomes ξ such that

$$X(\xi) \le x \text{ and } \xi \in m.$$

The properties of the conditional distribution $F_x(x \mid m)$ are similar to $F_X(x)$. For example,

$$F_X(\infty \mid m) = 1,\ F_X(-\infty \mid m) = 0,\ P\{x_1 < x \le x_2 \mid m\} = F_X(x_2 \mid m) - F_X(x_1 \mid m).$$

Definition: The conditional density of $X(\xi)$ given m is defined as

$$f_X(x \mid m) = \frac{dF_X(x \mid m)}{dx} = \lim_{\Delta x \to 0} \frac{P\{x \le X \le x + \Delta x \mid m\}}{\Delta x}.$$

$f_X(x \mid m)$ is non-negative and

$$\int_{-\infty}^{+\infty} f(x \mid m)\, dx = 1.$$

Expected Value and Moments

The expected value of a random variable $X(\xi)$ is defined as

$$E\{X\} = \int_{-\infty}^{+\infty} x f_X(x) dx = < X > .$$

For a discrete random variable with $f_X(x) = \sum_n P_n \delta(x - x_n)$

$$E\{X\} \approx \frac{x_1 + x_2 + \ldots + x_n}{n} .$$

Lebesgue Integral in sample space (Ensemble Average)

The mean of $X(\xi)$ may be written in terms of a Lebesgue integral in the sample space. i.e.,

$$E\{X\} = \int_{-\infty}^{+\infty} x f(x) dx = \sum_{i=-\infty}^{+\infty} x_i f(x_i) \Delta x_i = \sum_{i=-\infty}^{+\infty} x_i P\{x_i < X \le x_i + \Delta x_i\} = \int_S X dP .$$

Expected Value of g(X)

Definition: The expected values of a function of a random variable is defined as

$$E\{g(X)\} = \int_{-\infty}^{+\infty} g(x) f_X(x) dx .$$

When X is a discrete random variable,

$$E\{g(x)\} = \sum_i P_i g(x_i) .$$

Expected value is a linear operator. i.e.,

$$E\left\{\sum_{j=1}^{n} g_j(X)\right\} = \sum_{j=1}^{n} E\{g_j(x)\} .$$

Variance (σ^2)

Definition: The variance of a random variable is defined as

$$\sigma^2 = E\{x^2\} - \eta^2 .$$

Here, σ , is referred to as the standard deviation.

Moments

Definition: *kth* moment of a random variable, m_k , is defined as

$$m_k = E\{x^k\} = \int_{-\infty}^{+\infty} x^k f_X(x) dx, \; m_0 = 1, \; m_1 = \eta .$$

Definition: *kth* central moment of a random variable, μ_k , is defined as

$$\mu_k = E\{(x-\eta)^k\} = \int_{-\infty}^{+\infty} (x-\eta)^k f_X(x) dx .$$

$$\mu_0 = 1, \; \mu_1 = 0, \; \mu_2 = \sigma^2, \; \mu_3 = m_3 - 3\eta m_2 + 2\eta^3 .$$

Note that

$$\mu_k = E\{(x-\eta)^k\} = \sum_{i=0}^{k} \binom{k}{i} (-1)^i \eta^i m_{k-i} .$$

Moments of a Normal Random Variable

For a zero mean normal random variable with probability density function

$$f(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{x^2}{2\sigma^2}} ,$$

the moments are given as

$$E\{x^n\} = \begin{Bmatrix} 1\cdot 3\cdot \ldots \cdot (n-1)\sigma^n & n \; even \\ 0 & n \; odd \end{Bmatrix},$$

$$E\{|X|^n\} = \begin{Bmatrix} 1\cdot 3\cdot \ldots \cdot (n-1)\sigma^n & n \; even \\ \sqrt{\frac{2}{\pi}} 2^k k! \sigma^{2k+1} & n = 2k+1 \end{Bmatrix}.$$

Tchevycheff Inequality

For a random variable X with mean η and standard deviation σ,

$$P\{|X-\eta|\geq k\sigma\}\leq\frac{1}{k^2}$$

where k is a positive constant.

Proof:

$$\sigma^2=\int_{-\infty}^{+\infty}(x-\eta)^2 f(x)dx\geq\int_{|x-\eta|\geq k\sigma}(x-\eta)^2 f(x)dx$$
$$\geq k^2\sigma^2\int_{|x-\eta|\geq k\sigma}f(x)dx=k^2\sigma^2 P\{|x-\eta|\geq k\sigma\}.$$

Therefore,

$$P\{|X-\eta|\geq k\sigma\}\leq\frac{1}{k^2}.$$

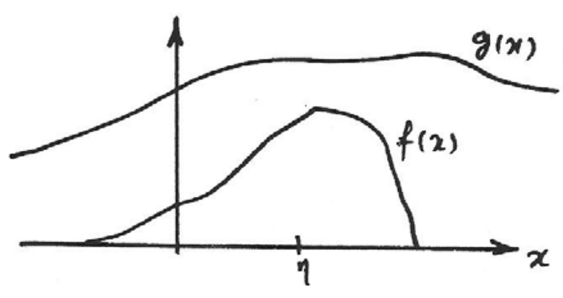
Approximate Evaluation of the Mean and Variance of g(X)

If $g(x)$ is a smoothly varying function then

$$E\{g(X)\}=\int_{-\infty}^{+\infty}g(x)f(x)dx\approx g(\eta)+g''(\eta)\frac{\sigma^2}{2},$$

and

$$\sigma_{g(x)}^2\approx g'^2(\eta)\sigma^2.$$



Here

$$\eta=E\{X\},$$

$$\sigma^2=E\{(X-\eta)^2\}.$$

The proof following by using a series expansion of the density function near it mean.