Boundary Layer

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Boundary Layer

Outline

- Flows Past Immersed Bodies
- Boundary Layer Flows (laminar)
- Blasius Solution
- Boundary Layer with Pressure Gradient
- Falkner-Scan Equation

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Boundary Layer

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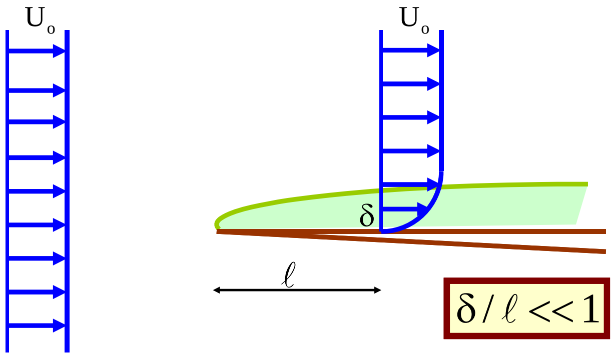
Boundary Layer over a Flat Plate



$\delta / \ell << 1$

Laminar Boundary Layer

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Boundary Layer Thickness

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Boundary Layer Thickness, δ → Distance at which $$\frac{u}{U_0} = 0.99$$

Displacement Thickness → $$\delta^* = \int_0^\infty \left(1 - \frac{u}{U_0}\right) dy$$

Momentum Thickness → $$\theta = \int_0^\infty \frac{u}{U_0}\left(1 - \frac{u}{U_0}\right) dy$$

Shape Factor → $$H = \frac{\delta^*}{\theta}$$

Boundary Layer Theory

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Steady Two-D Flows

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)$$

$$u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial y} + \nu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right)$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

Boundary Layer Theory

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Order of Magnitude Analysis

Prandtl

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

$$\frac{U_o}{\ell} \sim \frac{o\{v\}}{\delta}$$

→ $$o\{v\} \sim \frac{\delta U_o}{\ell}$$

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Order of Magnitude Analysis

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\left(\cancel{\frac{\partial^2 u}{\partial x^2}} + \frac{\partial^2 u}{\partial y^2}\right)$$

$$\frac{U_o^2}{\ell} \qquad \frac{o\{p\}}{\rho\ell} \qquad \nu\left(\frac{U_o}{\ell^2} \qquad \frac{U_o}{\delta^2}\right)$$

→ $$\delta \sim \sqrt{\frac{\nu\ell}{U_o}}$$ → $$\frac{\delta}{\ell} \sim \sqrt{\frac{\nu}{U_o\ell}} \sim \frac{1}{\sqrt{R_{e\ell}}}$$

Boundary Layer Theory

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Order of Magnitude Analysis

$$u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}=-\frac{\partial p}{\partial y}+\nu\left(\cancel{\frac{\partial^2 v}{\partial x^2}}+\frac{\partial^2 v}{\partial y^2}\right)$$

$$\delta\frac{U_o^2}{\ell^2} \qquad \frac{o\{p\}}{\rho\delta} \qquad \nu\left(\frac{\delta U_o}{\ell^3} \qquad \frac{U_o}{\delta\ell}\right)$$

$p\sim\rho U_o^2$ → $\frac{\partial p}{\partial y}\sim\frac{\delta^2}{\ell^2}\rho U_o^2\sim 0$

Boundary Layer Equations

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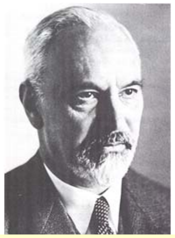
$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=-\frac{1}{\rho}\frac{dp}{dx}+\nu\frac{\partial^2 u}{\partial y^2}$$

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0$$



Ludwig Prandtl

Boundary Conditions →

at $y=0$ $u=0,\ v=0$

at $y=\infty$ $u=U_o$

Boundary Layer Equations

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$$-\frac{1}{\rho}\frac{dp}{dx}=U\frac{dU}{dx}$$

U (x) **is the external flow velocity**

$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=U\frac{dU}{dx}+\nu\frac{\partial^2 u}{\partial y^2}$$

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0$$

Boundary Layer over a Flat Plate

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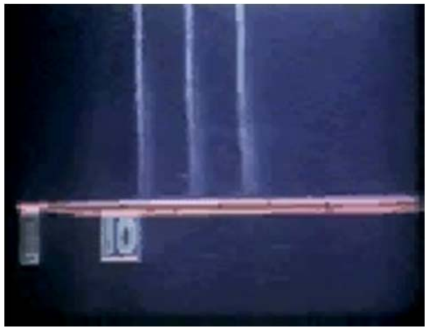

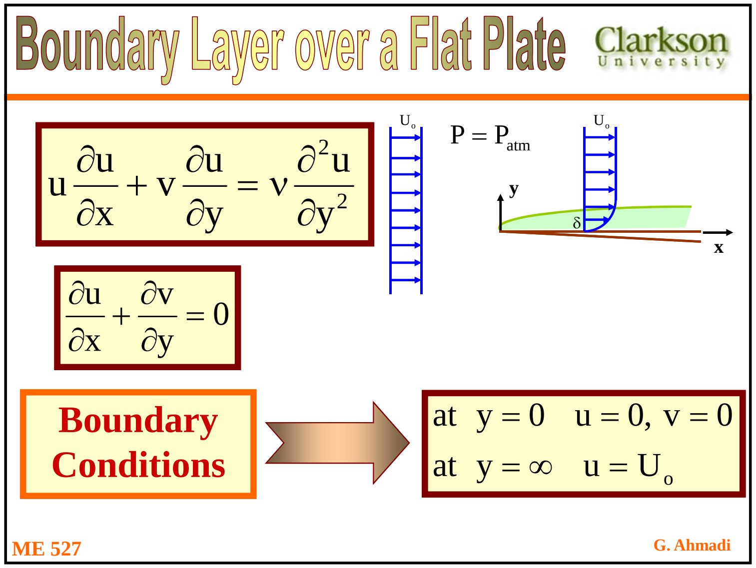

Boundary Layer over a Flat Plate

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$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=\nu\frac{\partial^2 u}{\partial y^2}$$

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0$$

Boundary Conditions

at $y = 0$ $\quad u = 0,\ v = 0$

at $y = \infty$ $\quad u = U_o$

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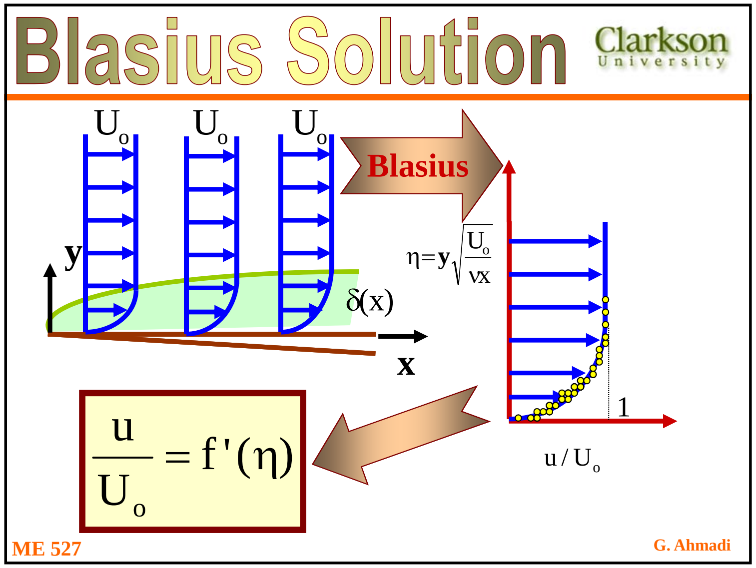

Blasius Solution

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$$\frac{u}{U_o}=f'(\eta)$$

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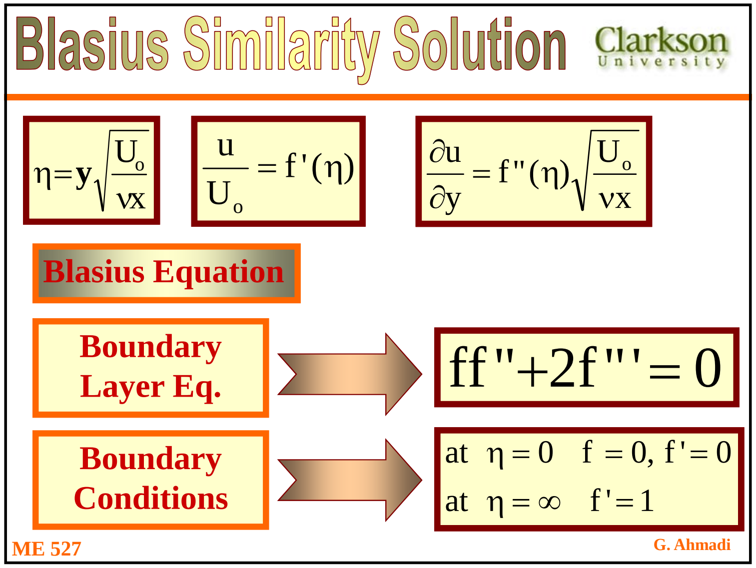

Blasius Similarity Solution

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$$\eta=y\sqrt{\frac{U_o}{\nu x}}$$

$$\frac{u}{U_o}=f'(\eta)$$

$$\frac{\partial u}{\partial y}=f''(\eta)\sqrt{\frac{U_o}{\nu x}}$$

Blasius Equation

Boundary Layer Eq.

$$ff''+2f'''=0$$

Boundary Conditions

at $\eta = 0$ $\quad f = 0,\ f' = 0$

at $\eta = \infty$ $\quad f' = 1$

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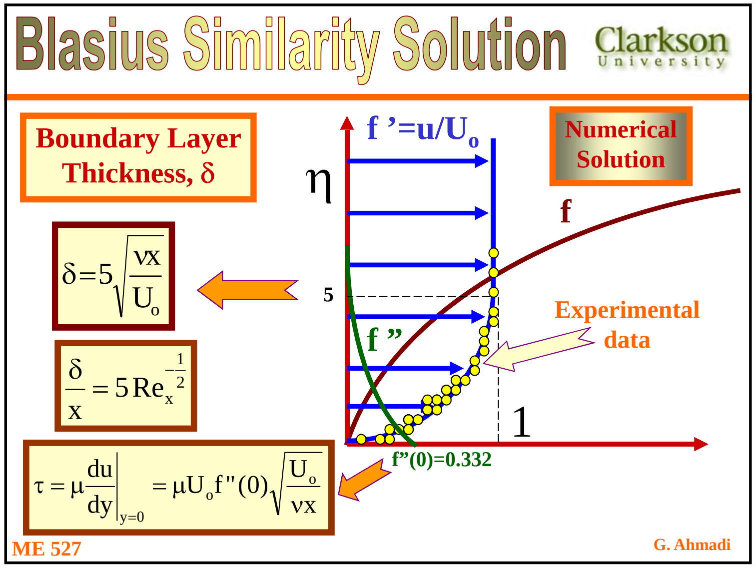

Blasius Similarity Solution

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Boundary Layer Thickness, δ

$$\delta=5\sqrt{\frac{\nu x}{U_o}}$$

$$\frac{\delta}{x}=5\,Re_x^{-\frac{1}{2}}$$

$$\tau=\mu\left.\frac{du}{dy}\right|_{y=0}=\mu U_o f''(0)\sqrt{\frac{U_o}{\nu x}}$$

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Boundary Layer over a Flat Plate

Friction Coefficient

$$C_F = \frac{\tau}{\frac{1}{2}\rho U_o^2} = \frac{2f''(0)}{\sqrt{R_{ex}}} = \frac{0.664}{\sqrt{R_{ex}}}$$

Drag Coefficient

$$C_D = \frac{D}{\frac{1}{2}\rho U_0^2 \ell} = \frac{4f''(0)}{\sqrt{R_{e\ell}}} = \frac{1.328}{\sqrt{R_{e\ell}}}$$

Boundary Layer over a Flat Plate

Displacement Thickness

$$\delta^* = \int_0^\infty \left(1 - \frac{U}{U_0}\right) dy = 1.721\sqrt{\frac{\nu x}{U_0}}$$

Momentum Thickness

$$\theta = \int_0^\infty \frac{U}{U_0}\left(1 - \frac{U}{U_0}\right) dy = 0.664\sqrt{\frac{\nu x}{U_0}}$$

Shape Factor

$$H = \frac{\delta^*}{\theta} = 2.51$$

Boundary Layer Equations with Pressure Gradient

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = U\frac{dU}{dx} + \nu\frac{\partial^2 u}{\partial y^2}$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

Stream Function, ψ

$$u = \frac{\partial \psi}{\partial y}$$

$$v = -\frac{\partial \psi}{\partial x}$$

Boundary Layer Equations with Pressure Gradient

Stream Function Formulation

$$\frac{\partial \psi}{\partial y}\frac{\partial^2 \psi}{\partial x \partial y} - \frac{\partial \psi}{\partial x}\frac{\partial^2 \psi}{\partial y^2} = U\frac{dU}{dx} + \nu\frac{\partial^3 \psi}{\partial y^3}$$

Boundary Conditions

$$\frac{\partial \psi}{\partial x} = \frac{\partial \psi}{\partial y} = 0 \quad at \quad y = 0$$

$$\frac{\partial \psi}{\partial y} = U(x) \quad at \quad y = \infty$$

Boundary Layer Equations with Pressure Gradient

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Similarity Solution

$y \sim x^a$

$\psi \sim x^b$

$U \sim x^m$

$$\underbrace{\frac{\partial\psi}{\partial y}\frac{\partial^2\psi}{\partial x\partial y}-\frac{\partial\psi}{\partial x}\frac{\partial^2\psi}{\partial y^2}}_{x^{2b-2a-1}}=\underbrace{U\frac{dU}{dx}}_{x^{2m-1}}+\underbrace{\nu\frac{\partial^3\psi}{\partial y^3}}_{x^{b-3a}}$$

Boundary Layer Equations with Pressure Gradient

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Hence for similarity solution

$2b-2a-1=2m-1=b-3a$

$b=(1+m)/2, \quad a=(1-m)/2$

For $U=u_1 x^m$

$$\eta=y\sqrt{\frac{m+1}{2}\frac{U}{\nu x}}=y\sqrt{\frac{m+1}{2}\frac{u_1}{\nu x^{1-m}}}$$

$$\psi=\sqrt{\frac{2\nu Ux}{m+1}}f(\eta)=\sqrt{\frac{2\nu u_1 x^{m+1}}{m+1}}f(\eta)$$

Boundary Layer Equations with Pressure Gradient

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$$u=Uf'(\eta)=u_1x^m f'(\eta)$$

$$v=-\sqrt{\frac{(m+1)\nu u_1 x^{m-1}}{2}}\left(f+\frac{m-1}{m+1}f'\right)$$

Falkner-Skan Equation

$$f'''+ff''+\beta(1-f'^2)=0$$

Boundary Layer Equations with Pressure Gradient

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Where

$$\beta=\frac{2m}{m+1}, \quad m=\frac{\beta}{2-\beta}$$

Boundary Conditions →

at $\eta=0$ $\quad f=0,\ f'=0$

at $\eta=\infty$ $\quad f'=1$

$\beta>0$, $m>0$, Accelerating Flow, No inflexion point

$\beta<0$, $m<0$, Decelerating Flow, with inflexion point

$\beta=-0.194$, $m=-0.092$ ~ Separation

Boundary Layer Equations with Pressure Gradient

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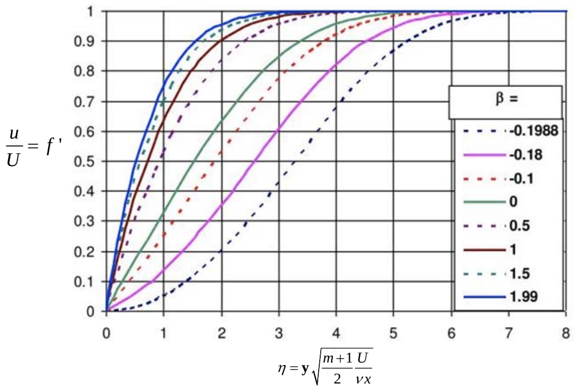

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Concluding Remarks

- Flows Past Immersed Bodies
- Boundary Layer Flows (Laminar)
- Prandtl Boundary Layer Theory
- Blasius Solution
- Boundary Layer with Pressure Gradient
- Falkner-Scan Similarity Solution

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Thank you!

Questions?