Advance Fluid Mechanics

Perturbation Theory

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Outline

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- **Algebraic Equations**
  - Regular Perturbation
  - Singular Perturbation
- **Differential Equations**
  - Regular Perturbation
  - Singular Perturbation

Perturbation- Algebraic

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Original Equation

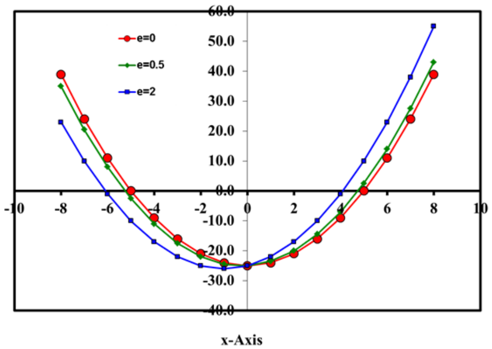
$$X^2 - 25 = 0$$

Perturbed equation

$$X^2 + \varepsilon X - 25 = 0$$

ε *Small parameter*

$$Y = X^2 + \varepsilon X - 25$$



Perturbation- Algebraic

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Perturbed equation $X^2 + \varepsilon X - 25 = 0$

Perturbation solution

$$X = X_0 + \varepsilon X_1 + \varepsilon^2 ...$$

$$(X_0 + \varepsilon X_1 + \varepsilon^2 ...)^2 + \varepsilon(X_0 + \varepsilon X_1 + \varepsilon^2 ...) - 25 = 0$$

$$X_0^2 - 25 + \varepsilon(2X_0X_1 + X_0) + \varepsilon^2 .... = 0$$

$$X_0^2 - 25 = 0, \quad X_0 = 5, -5$$

$$2X_0X_1 + X_0 = 0 \quad X_1 = 0.5$$

$$X = 5 - 0.5\varepsilon, \quad X = -5 - 0.5\varepsilon$$

Perturbation- Algebraic

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Perturbed Equation $X^2 + \varepsilon X - 25 = 0$

Perturbation Solution

$$X = 5 - 0.5\varepsilon, \quad X = -5 - 0.5\varepsilon$$

$$For \quad \varepsilon = 2, \quad X = 4, \quad X = -6$$

Exact Solution

$$X = \frac{-\varepsilon \pm \sqrt{\varepsilon^2 + 100}}{2}$$

$$For \quad \varepsilon = 2, \quad X = 4.1 \quad X = -6.1$$

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Singular Perturbation- Algebraic

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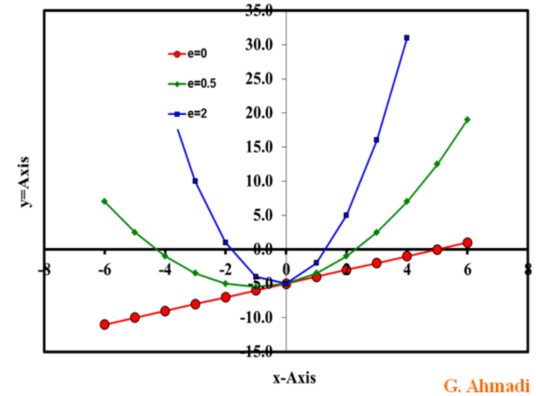
Original Equation

$$X - 5 = 0$$

Perturbed equation

$$\varepsilon X^2 + X - 5 = 0$$



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Singular Perturbation- Algebraic

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Perturbed Equation $\varepsilon X^2 + X - 5 = 0$

Exact Solution

$$X = \frac{-1 \pm \sqrt{1 + 20\varepsilon}}{2\varepsilon}$$

$$For \quad \varepsilon = 2, \quad X = 1.35 \quad X = -1.85$$

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Perturbation-Differential Equations

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Perturbation solution of differential equations with a small parameter, ε

Example $\frac{dy}{dx} + x = \varepsilon$ **BC** $y = 1, \; at \; x = 0$

Perturbation Solution $y = y_o + \varepsilon y_1 + \ldots$

$$\frac{dy_o}{dx} + \varepsilon \frac{dy_1}{dx} + x = \varepsilon$$

$$\varepsilon^o \rightarrow \quad \frac{dy_o}{dx} + x = 0, \quad y_o = -\frac{x^2}{2} + C = 1 - \frac{x^2}{2}$$

$$\varepsilon^1 \rightarrow \quad \frac{dy_1}{dx} = 1, \quad y_1 = x$$

$$y = 1 - \frac{x^2}{2} + \varepsilon x$$

Perturbation Solution, which is also exact solution

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Perturbation-Differential Equations

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Example $\frac{dy}{dx} - \varepsilon y = 1$ **ε = a small parameter**

BC $y = 0, \ at \ x = 0$

Solution $y = y_o + \varepsilon y_1 + \ldots$

$$\frac{dy_o}{dx} + \varepsilon \frac{dy_1}{dx} - \varepsilon(y_o + \varepsilon y_1) = 1$$

$\varepsilon^o \rightarrow$ $\frac{dy_o}{dx} = 1, \quad y_o = x + C = x$

$\varepsilon^1 \rightarrow$ $\frac{dy_1}{dx} = y_o = x, \quad y_1 = \frac{x^2}{2}$

$y = x + \varepsilon \frac{x^2}{2}$ **Perturbation Solution**

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For $\frac{dy}{dx} - \varepsilon y = 1$ **BC** $y = 0, \ at \ x = 0$

Exact Solution $y = \frac{1}{\varepsilon}\left(e^{\varepsilon x} - 1\right)$

$$y = \frac{1}{\varepsilon}\left(\left[1 + \varepsilon x + \frac{\varepsilon^2 x^2}{2} + \ldots\right] - 1\right) \quad for \ small \ \varepsilon$$

$$y = \frac{1}{\varepsilon}\left(\varepsilon x + \frac{\varepsilon^2 x^2}{2} + \ldots\right) = x\left(1 + \frac{\varepsilon x}{2} + \ldots\right)$$

$$y = x + \frac{\varepsilon x^2}{2} + \ldots$$

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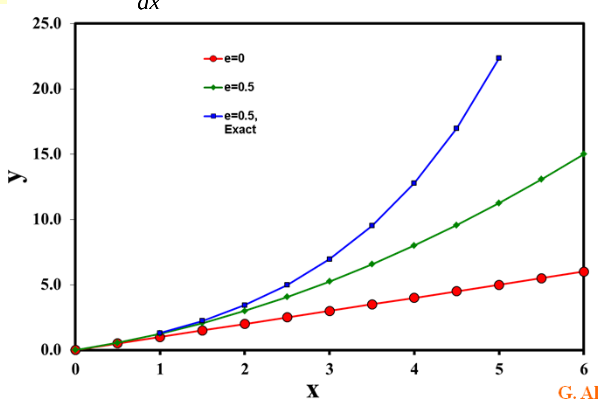
For $\frac{dy}{dx} - \varepsilon y = 1 \qquad y = 0, \ at \ x = 0$



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Singular Perturbation

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Matched Asymptotic Expansion

Given $\varepsilon \frac{d^2 y}{dx^2} + \frac{dy}{dx} = 0$ **BC** $y = 0, \ at \ x = 0$; $y = 1, \ at \ x = 1$

Exact Solution

$$y = \frac{1 - e^{\frac{-x}{\varepsilon}}}{1 - e^{\frac{-1}{\varepsilon}}}$$

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Outer Solution $\varepsilon \frac{d^2 y}{dx^2} + \frac{dy}{dx} = 0$

BC $y = 1, \ at \ x = 1$

$y_0 = 1, \ at \ x = 1$

$y_1 = 0, \ at \ x = 1$

Let $y = y_o + \varepsilon y_1 + ...$

$$\varepsilon \frac{d^2 y_o}{dx^2} + \varepsilon^2 \frac{d^2 y_1}{dx^2} + ... + \frac{dy_o}{dx} + \varepsilon \frac{dy_1}{dx} + ... = 0$$

$\varepsilon^0 \to \quad \frac{dy_o}{dx} = 0, \qquad y_o = C$

$\varepsilon^0 \to \quad \frac{dy_1}{dx} + \frac{d^2 y_o}{dx^2} = 0, \quad y_1 = C_1$

BC $C = 1 \qquad y_o = 1$

$C_1 = 0 \qquad y_1 = 0$

Outer Solution $y = y_o = 1$

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Inner Solution $\varepsilon \frac{d^2 y}{dx^2} + \frac{dy}{dx} = 0$

BC $y = 0, \ at \ x = 0$

$y = 1, \ at \ x = 1$

Let $X = \frac{x}{\sigma(\varepsilon)} \quad y(x,\varepsilon) = Y(X,\varepsilon)$

$$\frac{d^2 Y}{dX^2} + \frac{\sigma}{\varepsilon} \frac{dY}{dX} = 0$$

BC $Y = 0, \ at \ X = 0$

$Y = 1, \ at \ X = 1/\sigma$

Choose σ such that the singularity is minimum

$let \quad \sigma = \varepsilon$

$$\frac{d^2 Y}{dX^2} + \frac{dY}{dX} = 0$$

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Singular Perturbation

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Let $Y = Y_o + \varepsilon Y_1 + ...$

Then $\frac{d^2 Y_o}{dX^2} + \frac{dY_o}{dX} = 0$

BC $Y = 0, \ at \ X = 0$

$Y = 1, \ at \ X = \infty$

$Y_o = C_2 e^{-X} + C_3$

The inner solution must satisfy the inner boundary condition and match the inner limit of the outer solution, so

Inner Solution $Y_o = C_3 (1 - e^{-X})$

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Matched Asymptotic Expansion

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Matching

1. Inner limit of outer solution.

Write the outer solution in term of inner variable and then take limit as ε goes to zero

Outer Solution $y = 1$

As ε→0 $y = 1$

$y = 1$ **Inner limit of outer solution**

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2. Outer limit of inner solution.

Write the inner solution in term of outer variable and then take limit as ε goes to zero

Inner Solution

$$Y_o = C_3(1-e^{-X}) = C_3(1-e^{-x/\varepsilon})$$

As ε→0

$$Y_o = C_3$$

Outer limit of inner solution

3. Outer limit of inner solution = Inner limit of outer solution

Hence

$$C_3 = 1$$

$$Y_o = 1-e^{-X}$$

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4. Composite solution.

Composite solution = Inner Solution +Outer Solution – Inner Limit of Outer Solution

$$y_c = (1-e^{-x/\varepsilon}) + 1 - (1)$$

$$y_c = 1-e^{-x/\varepsilon}$$

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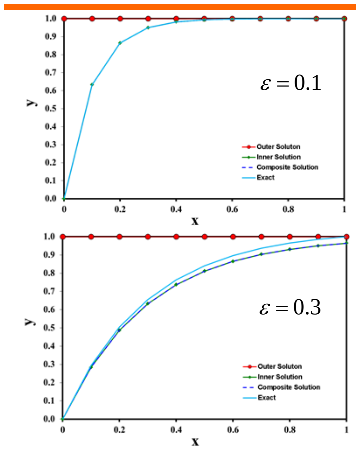

Composite Solution

$$y_c = 1-e^{-x/\varepsilon}$$

Exact Solution

$$y = \frac{1-e^{\frac{-x}{\varepsilon}}}{1-e^{\frac{-1}{\varepsilon}}}$$

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Given

$$\varepsilon\frac{d^2y}{dx^2} + (1+\varepsilon^2)\frac{dy}{dx} + (1-\varepsilon^2)y = 0$$

BC

$$y = \alpha, \quad at \ x = 0$$

$$y = \beta, \quad at \ x = 1$$

Exact Solution

$$y = e^{mx} \quad \to \quad \varepsilon m^2 + (1+\varepsilon^2)m + (1-\varepsilon^2) = 0$$

$$m = \frac{-(1+\varepsilon^2) \pm \sqrt{(1+\varepsilon^2)^2 - 4\varepsilon(1-\varepsilon^2)}}{2\varepsilon}$$

$$m = \frac{-(1+\varepsilon^2) \pm (1-\varepsilon^2 - 2\varepsilon)}{2\varepsilon} \quad \to \quad m = (1+\varepsilon), \quad m = -(\frac{1}{2\varepsilon} - 1)$$

$$y = \frac{[\beta - \alpha e^{-(\frac{1}{\varepsilon}-1)}]e^{-(1+\varepsilon)x} + [\alpha e^{-(1+\varepsilon)} - \beta]e^{-(\frac{1}{\varepsilon}-1)x}}{e^{-(1+\varepsilon)} - e^{-(\frac{1}{\varepsilon}-1)}}$$

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Outer Solution $\varepsilon\frac{d^2y}{dx^2}+(1+\varepsilon^2)\frac{dy}{dx}+(1-\varepsilon^2)y=0$

Let $y=y_o+\varepsilon y_1+\dots$

BC $y=\beta,\ \ at\ x=1$; $y_0=\beta,\ \ at\ x=1$; $y_1=0,\ \ at\ x=1$

$$\varepsilon\frac{d^2y_o}{dx^2}+\varepsilon^2\frac{d^2y_1}{dx^2}+\dots+(1+\varepsilon^2)(\frac{dy_o}{dx}+\varepsilon\frac{dy_1}{dx}+\dots)+(1-\varepsilon^2)(y_o+\varepsilon y_1+\dots)=0$$

$$\varepsilon^0\rightarrow\quad \frac{dy_o}{dx}+y_o=0,\quad\rightarrow\quad y_o=C\,e^{-x}=\beta\,e^{1-x}$$

$$\varepsilon^1\rightarrow\quad \frac{dy_1}{dx}+y_1=-\frac{d^2y_o}{dx^2}=-\beta\,e^{1-x},\quad\rightarrow\quad y_1=C_1e^{-x}-\beta\,xe^{1-x}$$

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Using $y_1=0,\ \ at\ x=1\ \rightarrow\ C_1=\beta e$

$y_1=\beta(1-x)e^{1-x}$

Then the Outer Solution is given as

$$y=y^o=y_0+\varepsilon y_1=\beta e^{1-x}+\varepsilon\beta(1-x)e^{1-x}$$

$$\rightarrow\quad y^o=\beta e^{1-x}[1+\varepsilon(1-x)+\varepsilon^2\dots]$$

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For $\varepsilon\rightarrow 0,\ 1/\varepsilon\rightarrow\infty$

Then the exact solution becomes

$$y=\beta e^{(1+\varepsilon)(1-x)}+\dots=\beta e^{(1-x)}e^{\varepsilon(1-x)}+\dots$$

$$y=\beta e^{(1-x)}[1+\varepsilon(1-x)+\frac{1}{2!}\varepsilon^2(1-x)^2+\dots]$$

$$y=\beta e^{(1-x)}+\varepsilon\beta(1-x)e^{(1-x)}+\dots$$

Which is consistent with the outer solution

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Original Equation

$$\varepsilon\frac{d^2y}{dx^2}+(1+\varepsilon^2)\frac{dy}{dx}+(1-\varepsilon^2)y=0$$

BC $y=\alpha,\ \ at\ x=0$; $y=\beta,\ \ at\ x=1$

Inner Solution

Let $X=\frac{x}{\sigma(\varepsilon)}=\frac{x}{\varepsilon}\ \rightarrow y(x,\varepsilon)=Y(X,\varepsilon)=Y^i(X,\varepsilon)$

Typically σ is selected such that the singularity is minimum

$$\frac{d^2Y^i}{dX^2}+(1+\varepsilon^2)\frac{dY^i}{dX}+\varepsilon(1-\varepsilon^2)Y^i=0$$

BC $Y^i=\alpha,\ \ at\ X=0$; $Y^i=\beta,\ \ at\ X=1/\varepsilon$

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Perturbation Solution

$$Y^i = Y_o + \varepsilon Y_1 + \dots$$

BC

$$Y_o = \alpha, \quad at \;\; X = 0$$
$$Y_1 = 0, \quad at \;\; X = 0$$

$$\frac{d^2Y_o}{dX^2} + \varepsilon \frac{d^2Y_1}{dX^2} + \dots + (1+\varepsilon^2)(\frac{dY_o}{dX} + \varepsilon \frac{dY_1}{dX} + \dots) + \varepsilon(1-\varepsilon^2)(Y_o + \varepsilon Y_1 + \dots) = 0$$

$$\varepsilon^0 \to \quad \frac{d^2Y_o}{dX^2} + \frac{dY_o}{dX} = 0, \quad \to \quad Y_o = A_o + B_o e^{-X}$$

$$\varepsilon^1 \to \quad \frac{d^2Y_1}{dX^2} + \frac{dY_1}{dX} = -Y_o$$

Using BC

$$A_o = \alpha - B_o$$
$$Y_o = \alpha - B_o + B_o e^{-X}$$

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Then

$$\frac{d^2Y_1}{dX^2} + \frac{dY_1}{dX} = -(\alpha - B_o) - B_o e^{-X}$$

BC

$$Y_1 = 0, \quad at \;\; X = 0$$

The solution becomes

$$Y_1 = A_1 + B_1 e^{-X} - (\alpha - B_o)X + B_o X e^{-X}$$
$$At \;\; X = 0, \; Y_1 = 0 \to A_1 = -B_1$$
$$Y_1 = -B_1 + B_1 e^{-X} - (\alpha - B_o)X + B_o X e^{-X}$$

Inner Solution

$$Y^i = \alpha - B_o + B_o e^{-X} + \varepsilon[-B_1 + B_1 e^{-X} - (\alpha - B_o)X + B_o X e^{-X}]$$

The constants B_0 and B_1 must be found by matching.

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Matching

The outer limit of (inner expansion) =
The inner limit of (outer expansion)

$$(y^o)^i = (Y^i)^o$$

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Matching

1. Inner limit of outer solution.
Write the outer solution in term of inner variable and then take limit as ε goes to zero

Outer Solution

$$y^o = \beta e^{1-x}[1 + \varepsilon(1-x) + \varepsilon^2 \dots]$$

Outer Solution in term of inner variable

$$y^o = \beta e^{1-\varepsilon X}[1 + \varepsilon(1 - \varepsilon X)]$$

As ε→0

$$(y^o)^i = \beta e[1 + \varepsilon(1 - X)]$$

Inner limit of outer solution

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2. Outer limit of inner solution.

Write the inner solution in term of outer variable and then take limit as ε goes to zero

Inner Solution

$$Y^i = \alpha - B_o + B_o e^{-X} + \varepsilon[-B_1 + B_1 e^{-X} - (\alpha - B_o)X + B_o X e^{-X}]$$

Inner Solution in term of outer variable

$$Y^i = \alpha - B_o + B_o e^{-x/\varepsilon} + \varepsilon[-B_1 + B_1 e^{-x/\varepsilon} - (\alpha - B_o)\frac{x}{\varepsilon} + B_o \frac{x}{\varepsilon} e^{-x/\varepsilon}]$$

As ε→0

$$(Y^i)^o = \alpha - B_o + [-\varepsilon B_1 - (\alpha - B_o)x]$$

$$(Y^i)^o = (\alpha - B_o)(1-x) - \varepsilon B_1$$

Outer limit of inner solution

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3. Matching

Inner limit of outer solution = Outer limit of inner solution

$$(y^o)^i = \beta e[1 + \varepsilon(1 - X)] = (\alpha - B_o)(1-x) - \varepsilon B_1 = (Y^i)^o$$

$$\beta e[1 - x + \varepsilon] = (\alpha - B_o)(1-x) - \varepsilon B_1$$

Hence

$$B_o = (\alpha - \beta e)$$

$$B_1 = -\beta e$$

Inner Solution

$$Y^i = \beta e + (\alpha - \beta e)e^{-X} + \varepsilon[\beta e(1 - e^{-X}) - \beta e X + (\alpha - \beta e)X e^{-X}]$$

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4. Composite solution.

Composite solution = Inner Solution +Outer Solution – Inner Limit of Outer Solution

$$y_c = y^o + y^i - (y^o)^i$$

$$y_c = \beta e + (\alpha - \beta e)e^{-X} + \varepsilon[\beta e(1 - e^{-X}) - \beta e X + (\alpha - \beta e)X e^{-X}]$$
$$+ \beta e^{1-x}[1 + \varepsilon(1-x) + \varepsilon^2 \ldots] - \beta e[1 + \varepsilon(1 - X)]$$

Composite solution

$$y_c = (\alpha - \beta e)e^{-x/\varepsilon}(1+x) + \beta e^{1-x} + \varepsilon\beta e[-e^{-x/\varepsilon} + e^{-x}(1-x)]$$

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For $\alpha = 0, \quad \beta = 1$

Outer Solution $y^o = e^{1-x}[1 + \varepsilon(1-x)]$

Inner Solution

$$Y^i = e[1 - x - (1+x)e^{-x/\varepsilon}] + \varepsilon e(1 - e^{-x/\varepsilon})$$

Composite Solution

$$y_c = -e^{1-x/\varepsilon}(1+x) + e^{1-x} + \varepsilon[-e^{1-x/\varepsilon} + e^{1-x}(1-x)]$$

Exact Solution

$$y = \frac{e^{-(1+\varepsilon)x} - e^{-(\frac{1}{\varepsilon}-1)x}}{e^{-(1+\varepsilon)} - e^{-(\frac{1}{\varepsilon}-1)}}$$

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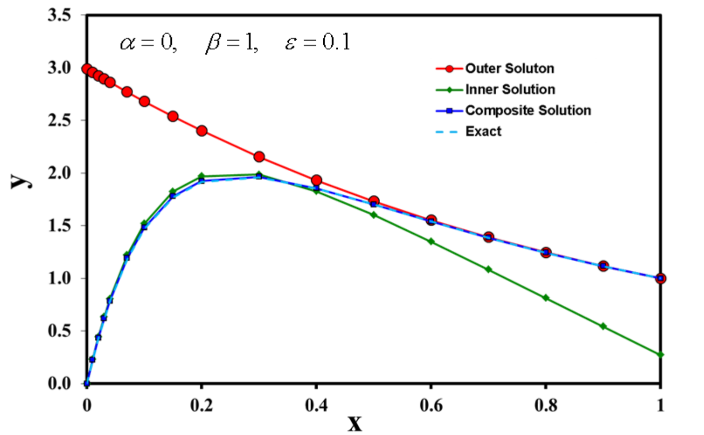

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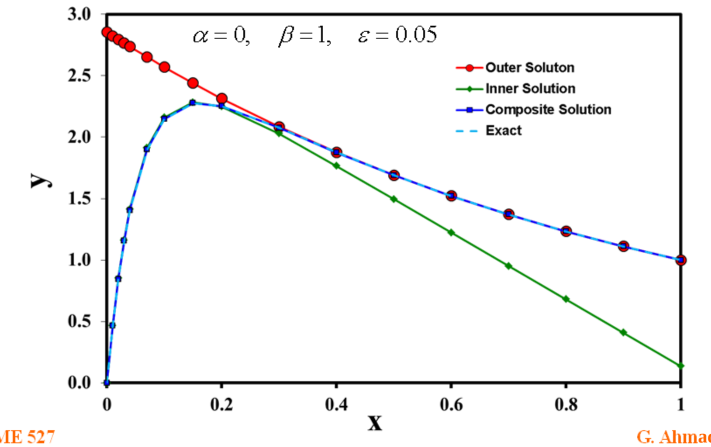

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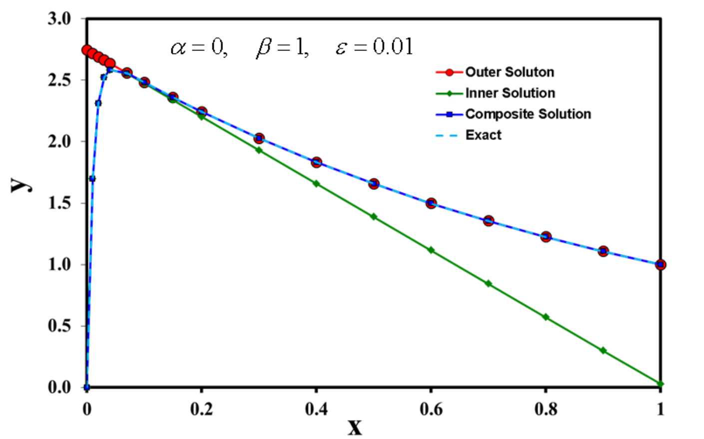

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Singular Perturbation

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Given

$$\varepsilon \frac{d^2y}{dx^2} + 2x\frac{dy}{dx} - 2xy = 0$$

$$y = 0, \quad at \ x = 0$$
$$y = 1, \quad at \ x = 1$$

Outer Solution

$$y^o = y_o + \varepsilon y_1 + \ldots$$

$$y = 1, \quad at \ x = 1$$
$$y_0 = 1, \quad at \ x = 1$$
$$y_1 = 0, \quad at \ x = 1$$

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Outer Solution

$$\varepsilon\frac{d^2y_o}{dx^2}+\varepsilon^2\frac{d^2y_1}{dx^2}+\dots+2x(\frac{dy_o}{dx}+\varepsilon\frac{dy_1}{dx}+\dots-y_o-\varepsilon y_1-\dots)=0$$

$$\frac{dy_o}{dx}-y_o=0,\qquad y_o=C\,e^x\qquad C=e\qquad y_o=e^{x-1}$$

$$\frac{dy_1}{dx}-y_1=-\frac{1}{2x}\frac{d^2y_o}{dx^2}$$

Outer Solution $y^o=y_o=e^{x-1}$

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Inner Solution $\varepsilon\frac{d^2y}{dx^2}+2x\frac{dy}{dx}-2xy=0$ **BC** $y=0,\ at\ x=0$; $y=1,\ at\ x=1$

Let $X=\frac{x}{\sigma(\varepsilon)},\quad y(x,\varepsilon)=Y(X,\varepsilon)$

$$\frac{\varepsilon}{\sigma^2}\frac{d^2Y}{dX^2}+2X\frac{dY}{dX}-2\sigma XY=0$$

$$\frac{d^2Y}{dX^2}+\frac{2\sigma^2X}{\varepsilon}\frac{dY}{dX}-\frac{2\sigma^3}{\varepsilon}XY=0$$

BC $Y=0,\ at\ X=0$; $Y=1,\ at\ X=1/\sigma$

Choose σ such that the singularity is minimum

$$let\quad \sigma=\varepsilon^{1/2},\quad X=\frac{x}{\varepsilon^{1/2}}$$

$$\frac{d^2Y}{dX^2}+2X\frac{dY}{dX}-2\varepsilon^{1/2}XY=0$$

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Let $Y=Y_o+\varepsilon^{1/2}Y_1+\dots$

Then $\frac{d^2Y_o}{dX^2}+2X\frac{dY_o}{dX}=0$ **BC** $Y=0,\ at\ X=0$; $Y=1,\ at\ X=\infty$

$$\frac{d^2Y_o/dX^2}{dY_o/dX}=-2X,\qquad \frac{dY_o}{dX}=Ce^{-X^2}$$

$$Y_o=C\int_0^X e^{-X_1^2}dX_1=C_1\,erf(X),\qquad where\ Y_o(0)=0$$

The inner solution must satisfy the inner boundary condition and match the inner limit of the outer solution, so

Inner Solution $y^i=Y_o=C_1\,erf(X)$

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Matched Asymptotic Expansion Clarkson University

Matching

1. Inner limit of outer solution.
Write the outer solution in term of inner variable and then take limit as ε goes to zero

Outer Solution

As ε→0 $y^o=e^{x-1},\quad (y^o)^i=\lim_{\varepsilon\to 0}e^{\varepsilon X-1}=e^{-1}$

Inner limit of outer solution

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2. Outer limit of inner solution.

Write the inner solution in term of outer variable and then take limit as ε goes to zero

Inner Solution

$$y^i = Y_o = C_1 erf(X), \quad (y^i)^o = \lim_{\varepsilon \to 0} C_1 erf(\frac{x}{\varepsilon^{1/2}}) = C_1$$

Outer limit of inner solution

3. Outer limit of inner solution = Inner limit of outer solution

Hence

$$C_1 = e^{-1}$$

$$y^i = Y_o = e^{-1} erf(X)$$

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4. Composite solution.

Composite solution = Inner Solution +Outer Solution – Inner Limit of Outer Solution

$$y_c = y^o + y^i - (y^o)^i$$

$$y^o = y_o = e^{x-1} \qquad y^i = Y_o = e^{-1} erf(X)$$

$$y_c = e^{x-1} + e^{-1} erf(x/\varepsilon^{1/2}) - e^{-1}$$

$$y_c = e^{-1}[e^x + erf(x/\varepsilon^{1/2}) - 1]$$

$$y_c = e^{-1}[e^x - erfc(x/\varepsilon^{1/2})]$$

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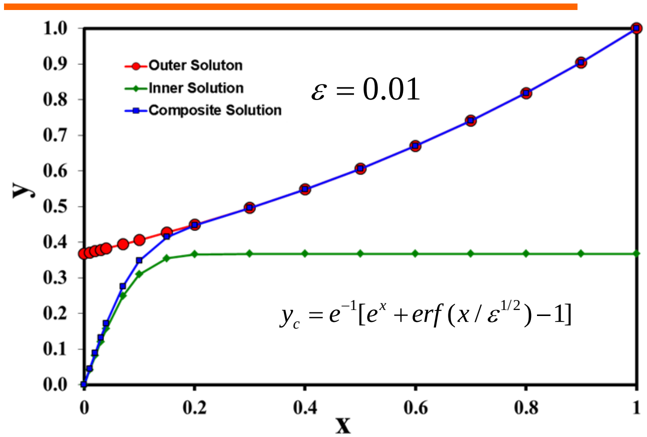

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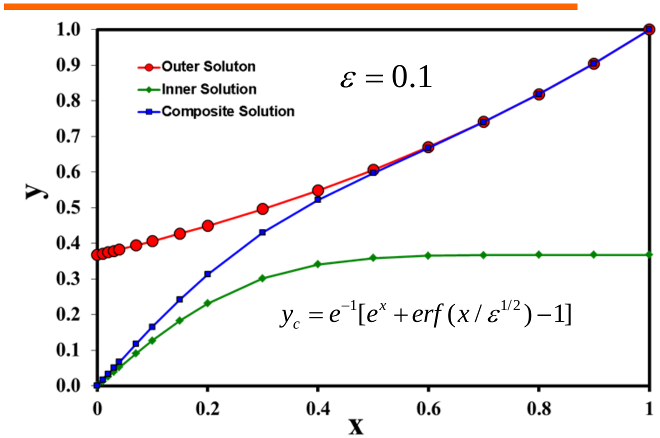

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