ME 527 – Advanced Fluids

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Exact Solutions to the Navier-Stokes Equation

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Exact Solutions

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Outline

- Plate Suddenly Set in Motion
- Oscillating Plate
- Unsteady Pipe Flows
- Steady Flows in Noncircular Pipes
- Elliptic Cross Section Pipes
- Triangular Cross-Section Pipes

Plate Suddenly Set in Motion

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$$\frac{\partial u}{\partial t} = \nu \frac{\partial^2 u}{\partial y^2}$$

B.C.

$y = 0 \rightarrow u = U_0$

$y = \infty \rightarrow u = 0$

I.C.

$t = 0 \rightarrow u = 0$

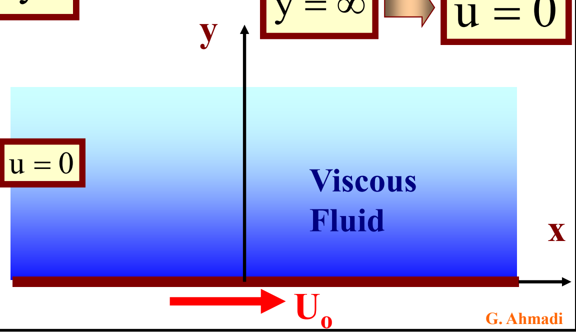

Plate Suddenly Set in Motion

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Similarity Solution

Let $t \sim t^1$ $\quad y \sim t^a$

Navier-Stokes $\rightarrow 1 = 2a \rightarrow a = 1/2$

Similarity Variables $\rightarrow \eta = \frac{y}{2\sqrt{\nu t}}$ $\quad \frac{u}{U_0} = f(\eta)$

Plate Suddenly Set in Motion

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NS → $f'' + 2\eta f' = 0$

→ $f' = c e^{-\eta^2}$

B.C. → $f(0) = 1$, $f(\infty) = 0$

$$f = 1 - \frac{2}{\sqrt{\pi}} \int_0^{\eta} e^{-\eta_1^2} d\eta_1 = 1 - \operatorname{erf}(\eta)$$

Solution → $u = U_0 \operatorname{erfc}\left(\frac{y}{2\sqrt{\nu t}} \right)$

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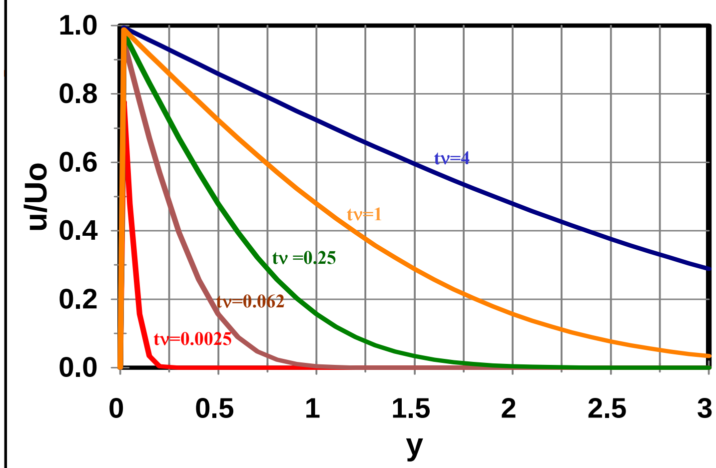

Variation of velocity profile with time.

Oscillating Plate

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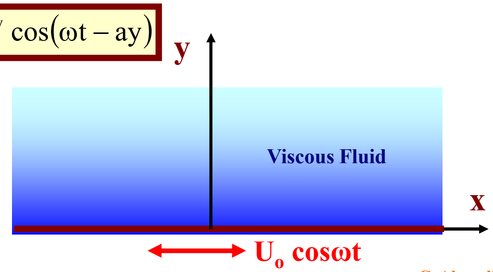
$$\frac{\partial u}{\partial t} = \nu \frac{\partial^2 u}{\partial y^2}$$

B.C. $y = 0$ → $u = U_0 \cos \omega t$

$y = \infty$ → $u = 0$

Let $u = U_0 e^{-ky} \cos(\omega t - ay)$



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Oscillating Plate

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$$\frac{\partial u}{\partial t} = -\omega U_0 e^{-ky} \sin(\omega t - ay)$$

$$\frac{\partial u}{\partial y} = U_0 e^{-ky} \left(-k \cos(\omega t - ay) + a \sin(\omega t - ay) \right)$$

Navier-Stokes Equation

$$-\omega \sin\theta = \nu\left(\left(k^2 - a^2\right)\cos\theta - 2ak \sin\theta \right)$$

Matching → $a^2 = k^2$, $\omega = 2ak\nu = 2k^2\nu$

Solution $u = U_0 e^{-ky} \cos(\omega t - ky)$, $k = \sqrt{\frac{\omega}{2\nu}}$

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Unsteady Flow in a Tube
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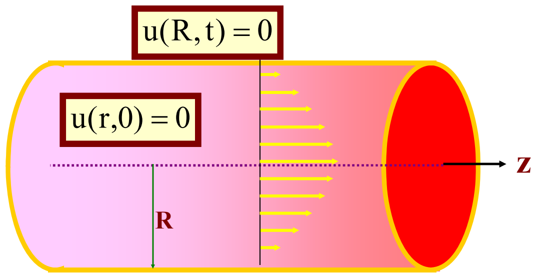
Navier Stokes →

$$\frac{\partial v_z}{\partial t} = -\frac{1}{\rho}\frac{dP}{dz} + \nu \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_z}{\partial r}\right)$$



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Nondimensional Variables →

$$\xi = \frac{r}{R} \qquad \tau = \frac{\mu t}{\rho R^2} = \frac{\nu t}{R^2}$$

$$v_z = -\frac{1}{4\mu}\frac{dP}{dz}R^2\varphi(\xi)$$

Navier- Stokes →

$$\frac{\partial \varphi}{\partial \tau} = 4 + \frac{1}{\xi}\frac{\partial}{\partial \xi}\left(\xi\frac{\partial \varphi}{\partial \xi}\right)$$

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Boundary Conditions →

$$\xi = 1 \qquad \varphi = 0$$

$$\tau = 0 \qquad \varphi = 0$$

Changing Variable →

$$\phi = 1 - \xi^2 - \psi$$

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Navier Stokes →

$$\frac{\partial \psi}{\partial \tau} = \frac{1}{\xi}\frac{\partial}{\partial \xi}\left(\xi\frac{\partial \psi}{\partial \xi}\right)$$

Boundary Conditions →

$$\xi = 1 \qquad \psi = 0$$

$$\tau = 0 \qquad \psi = 1 - \xi^2$$

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Separation of Variables

$$\psi = F(\xi)T(\tau) \Rightarrow \frac{\dot{T}}{T} = \frac{1}{F\xi}\frac{d}{d\xi}\left(\xi\frac{dF}{d\xi}\right) = -\alpha^2$$

$$\dot{T} + \alpha^2 T = 0 \Rightarrow T = Ce^{-\alpha^2\tau}$$

Bessel Equation

$$\xi^2\frac{d^2F}{d\xi^2} + \xi\frac{dF}{d\xi} + \alpha^2\xi^2F = 0$$

Bessel Functions

$$F = AJ_0(\alpha\xi) + BY_0(\alpha\xi)$$

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Boundary Conditions

$$Y_0(0) \to \infty \qquad F(0) \sim \text{finite} \Rightarrow B = 0$$

$$F(1) = 0 \Rightarrow J_0(\alpha) = 0$$

Eigenvalues

$$\alpha_1 = 2.405 \qquad \alpha_2 = 5.52 \qquad \alpha_3 = 8.654$$

General Solution

$$\psi = \sum_n A_n e^{-\alpha_n^2\tau} J_0(\alpha_n\xi)$$

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Initial Condition

$$1 - \xi^2 = \sum_n A_n J_0(\alpha_n\xi)$$

$$A_n = \frac{\int_0^1 (1-\xi^2)\xi J_0(\alpha_n\xi)d\xi}{\int_0^1 \xi J_0^2(\alpha_n\xi)d\xi} = \frac{8}{\alpha_n^3 J_1^2(\alpha_n)}$$

Solution

$$\psi = 8\sum_n \frac{e^{-\alpha_n^2\tau} J_0(\alpha_n\xi)}{\alpha_n^3 J_1(\alpha_n)}$$

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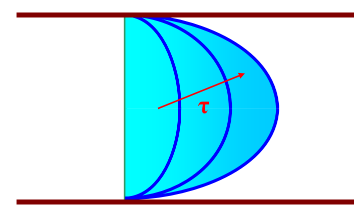
Solution

$$\varphi = 1 - \xi^2 - 8\sum_n \frac{J_0(\alpha_n\xi)}{\alpha_n^3 J_1(\alpha_n)} e^{-\alpha_n\tau}$$



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Noncircular Pipe Flows

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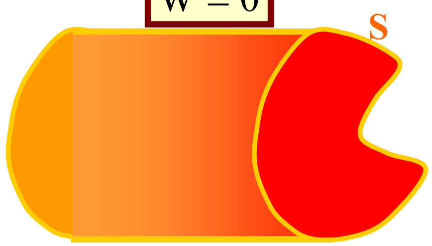
Navier Stokes → $\nabla^2 W = \frac{1}{\mu}\frac{dP}{dz} = \text{const}$

$W = 0$



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Elliptical Pipes

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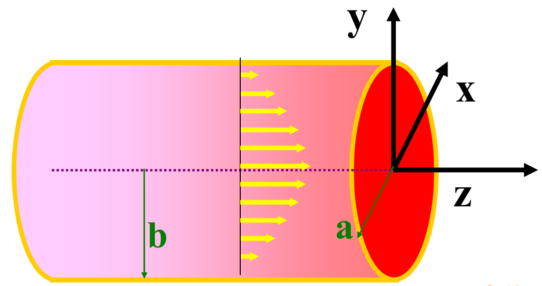
Ellipse

$\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1$ **Let** $w = A\left(1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}\right)$



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Elliptical Pipes

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NS → $\nabla^2 w = -A\left(\frac{2}{a^2} + \frac{2}{b^2}\right) = -\frac{2A(a^2+b^2)}{a^2 b^2} = \frac{1}{\mu}\frac{dP}{dz}$

$$w = -\frac{1}{2\mu}\frac{dP}{dz}\frac{a^2 b^2}{a^2+b^2}\left(1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}\right)$$

Flow Rate

$Q = \iint w\,dx\,dy$ $Q = -\frac{\pi}{4\mu}\frac{dP}{dz}\frac{a^3 b^3}{a^2+b^2}$

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Triangular Pipes

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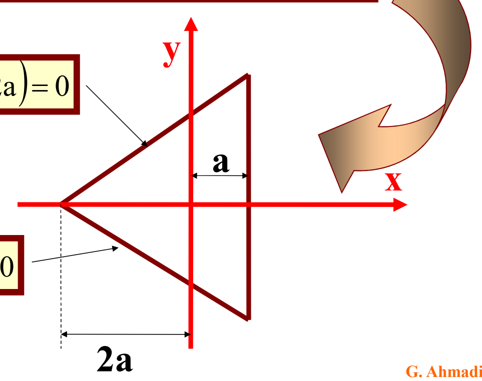
$$f(x,y) = (x-a)\left(x-\sqrt{3}y+2a\right)\left(x+\sqrt{3}y+2a\right) = 0$$

$\left(x-\sqrt{3}y+2a\right) = 0$

$\left(x+\sqrt{3}y+2a\right) = 0$



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Triangular Pipes

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Let $w = Af(x,y)$

NS $\nabla^2 w = A\nabla^2 f(x,y) = 12aA = \frac{1}{\mu}\frac{dP}{dz}$

$A = \frac{1}{12\mu a}\frac{dP}{dx}$

Solution

$$w = \frac{1}{12\mu a}\frac{dP}{dx}(x-a)(x-\sqrt{3}y+2a)(x+\sqrt{3}y+2a)$$

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Concluding Remarks

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Thank you!

Questions?

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