ME 326 - Intermediate Fluid Mechanics

Clarkson University

Boundary Layer

Goodarz Ahmadi

Department of Mechanical and Aeronautical Engineering

Clarkson University

Potsdam, NY 13699-5725

ME 326 G. Ahmadi

Boundary Layer

Clarkson University

Outline

- Flows Past Immersed Bodies
- Boundary Layer Flows (laminar)
- Blasius Solution
- Momentum Integral Method
- Turbulent Boundary Layer Flows

ME 326 G. Ahmadi

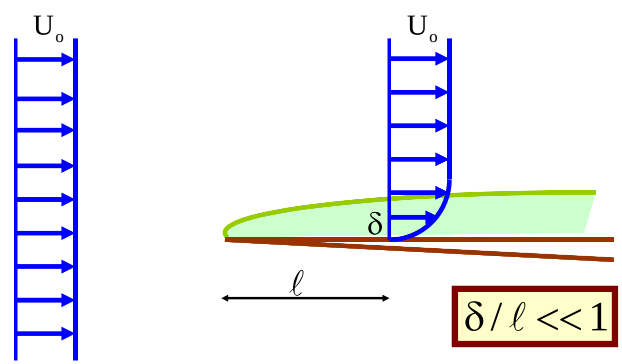
Boundary Layer

Clarkson University

ME 326 G. Ahmadi

Boundary Layer over a Flat Plate

Clarkson University



Laminar Boundary Layer

ME 326 G. Ahmadi

Boundary Layer Thickness

Clarkson University

Boundary Layer Thickness, δ → Distance at which $\frac{u}{U_0} = 0.99$

Displacement Thickness → $\delta^* = \int_0^\infty \left(1 - \frac{u}{U_0}\right) dy$

Momentum Thickness → $\theta = \int_0^\infty \frac{u}{U_0}\left(1 - \frac{u}{U_0}\right) dy$

Shape Factor → $H = \frac{\delta^*}{\theta}$

ME 326 G. Ahmadi

Boundary Layer Theory

Clarkson University

Steady Two-D Flows

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)$$

$$u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial y} + \nu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right)$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

ME 326 G. Ahmadi

Boundary Layer Theory

Clarkson University

Order of Magnitude Analysis

Prandtl

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

$$\frac{U_o}{\ell} \sim \frac{o\{v\}}{\delta}$$

→ $o\{v\} \sim \frac{\delta U_o}{\ell}$

ME 326 G. Ahmadi

Boundary Layer Theory

Clarkson University

Order of Magnitude Analysis

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)$$

$$\frac{U_o^2}{\ell} \qquad \frac{o\{p\}}{\rho\ell} \qquad \nu\left(\frac{U_o}{\ell^2} \qquad \frac{U_o}{\delta^2}\right)$$

→ $\delta \sim \sqrt{\frac{\nu\ell}{U_o}}$ → $\frac{\delta}{\ell} \sim \sqrt{\frac{\nu}{U_o\ell}} \sim \frac{1}{\sqrt{R_{e\ell}}}$

ME 326 G. Ahmadi

Boundary Layer Theory

Clarkson University

Order of Magnitude Analysis

$$u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}=-\frac{\partial p}{\partial y}+\nu\left(\cancel{\frac{\partial^2 v}{\partial x^2}}+\frac{\partial^2 v}{\partial y^2}\right)$$

$$\delta\frac{U_o^2}{\ell^2}\qquad\frac{o\{p\}}{\rho\delta}\qquad\nu\left(\frac{\delta U_o}{\ell^3}\qquad\frac{U_o}{\delta\ell}\right)$$

$$p\sim\rho U_o^2\quad\Rightarrow\quad\frac{\partial p}{\partial y}\sim\frac{\delta^2}{\ell^2}\rho U_o^2\sim 0$$

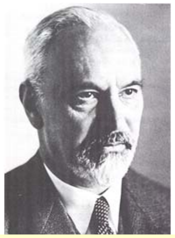
ME 326 G. Ahmadi

Boundary Layer Equations

Clarkson University

$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=-\frac{1}{\rho}\frac{dp}{dx}+\nu\frac{\partial^2 u}{\partial y^2}$$

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0$$



Ludwig Prandtl

Boundary Conditions →

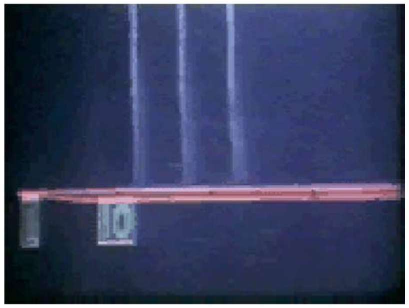
at $y=0$ $u=0,\ v=0$

at $y=\infty$ $u=U_o$

ME 326 G. Ahmadi

Boundary Layer over a Flat Plate

Clarkson University



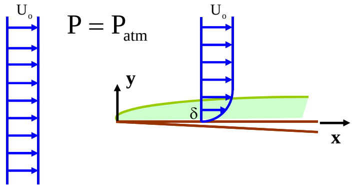
ME 326 G. Ahmadi

Boundary Layer over a Flat Plate

Clarkson University

$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=\nu\frac{\partial^2 u}{\partial y^2}$$

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0$$



Boundary Conditions →

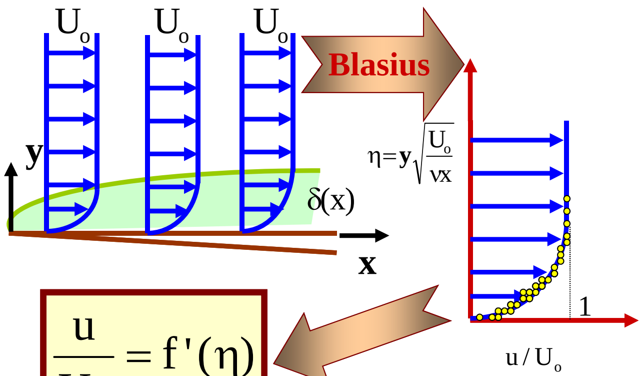
at $y=0$ $u=0,\ v=0$

at $y=\infty$ $u=U_o$

ME 326 G. Ahmadi

Blasius Solution

Clarkson University



$$\frac{u}{U_o} = f'(\eta)$$

ME 326 G. Ahmadi

Blasius Similarity Solution

Clarkson University

$$\eta = y\sqrt{\frac{U_o}{\nu x}} \qquad \frac{u}{U_o} = f'(\eta) \qquad \frac{\partial u}{\partial y} = f''(\eta)\sqrt{\frac{U_o}{\nu x}}$$

Blasius Equation

Boundary Layer Eq. → $ff'' + 2f''' = 0$

Boundary Conditions → at $\eta = 0$ $f = 0, f' = 0$; at $\eta = \infty$ $f' = 1$

ME 326 G. Ahmadi

Blasius Similarity Solution

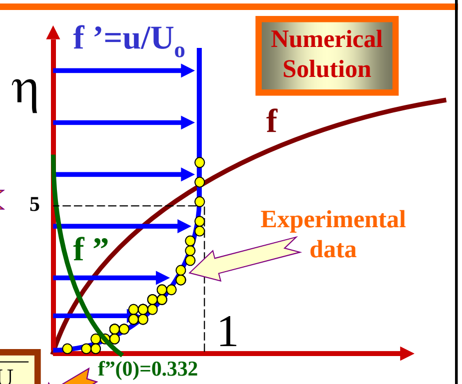
Clarkson University

Boundary Layer Thickness, δ

$$\delta = 5\sqrt{\frac{\nu x}{U_o}}$$

$$\frac{\delta}{x} = 5\,Re_x^{-\frac{1}{2}}$$

$$\tau = \mu \frac{du}{dy}\bigg|_{y=0} = \mu U_o f''(0)\sqrt{\frac{U_o}{\nu x}}$$



ME 326 G. Ahmadi

Boundary Layer over a Flat Plate

Clarkson University

Friction Coefficient

$$C_F = \frac{\tau}{\frac{1}{2}\rho U_o^2} = \frac{2f''(0)}{\sqrt{R_{ex}}} = \frac{0.664}{\sqrt{R_{ex}}}$$

Drag Coefficient

$$C_D = \frac{D}{\frac{1}{2}\rho U_o^2 \ell} = \frac{4f''(0)}{\sqrt{R_{e\ell}}} = \frac{1.328}{\sqrt{R_{e\ell}}}$$

ME 326 G. Ahmadi

Boundary Layer over a Flat Plate

Clarkson University

Displacement Thickness → $\delta^* = \int_0^\infty \left(1 - \frac{U}{U_0}\right) dy = 1.721\sqrt{\frac{\nu x}{U_0}}$

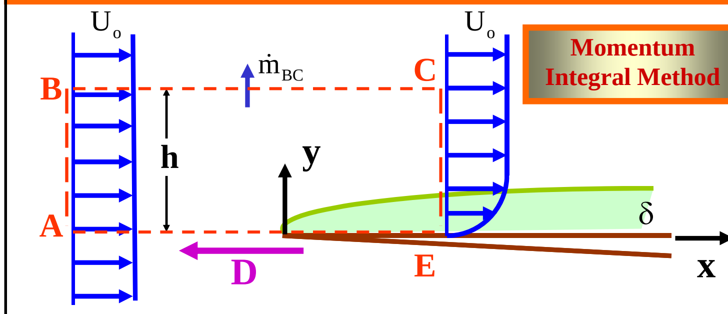
Momentum Thickness → $\theta = \int_0^\infty \frac{U}{U_0}\left(1 - \frac{U}{U_0}\right) dy = 0.664\sqrt{\frac{\nu x}{U_0}}$

Shape Factor → $H = \frac{\delta^*}{\theta} = 2.51$

ME 326 G. Ahmadi

Boundary Layer over a Flat Plate

Clarkson University



Momentum Integral Method

Conservation of Mass

$\rho\int_0^h u\,dy + \dot{m}_{BC} - \rho\int_0^h U_o\,dy = 0$

$\dot{m}_{BC} = \rho\int_0^h (U_o - u)\,dy = \rho U_o \delta^*$

ME 326 G. Ahmadi

Integral Method

Clarkson University

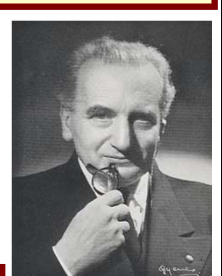
Momentum Equation

$\sum_{Outlets} \rho_o V_o A_o V_{ox} - \sum_{Inlets} \rho_i V_i A_i V_{ix} = \sum F_x$

$\rho\int_0^h u^2 dy + \dot{m}_{BC} U_o - \rho\int_0^h U_o^2 dy = -D$

$D = \rho\int_0^h u(U_o - u)\,dy = \rho U_o^2 \theta$

von Karman Momentum Integral → $\tau_w = \rho U_o^2 \frac{d\theta}{dx}$



von Karman

ME 326 G. Ahmadi

Integral Method

Clarkson University

Procedure

- **Assume a velocity profile that satisfies the boundary conditions.**
- **Evaluate wall shear stress and θ.**
- **Use Momentum Integral and find δ**
- **Evaluate Boundary Layer parameters θ, δ, C_F, C_D.**

ME 326 G. Ahmadi

Boundary Layer

Clarkson University

Concluding Remarks

- Flows Past Immersed Bodies
- Boundary Layer Flows (Laminar)
- Prandtl Boundary Layer Theory
- Blasius Solution
- Momentum Integral Method
- Turbulent Boundary Layer Flows

ME 326 G. Ahmadi

Clarkson University

Thank you!

Questions?

ME 326 G. Ahmadi