The Karhunen-Loeve Orthagonal Expansion

Suppose $\varphi_n(t)$ are a set of orthonormal basis in the interval (0,T). A function X(t) (deterministic or random) may be expanded as

$$X(t) = \sum c_n \varphi_n(t), \qquad 0 < t < T, \tag{1}$$

where the coefficient c_n are given by

$$c_n = \int_0^T X(t)\varphi_n(t)dt. \tag{2}$$

Note that the property

$$\int_0^T \varphi_n(t)\varphi_m^*(t)dt = \delta_{nm} \tag{3}$$

was used in the derivation of (2).

When X(t) is a random function, the coefficients c_n become random coefficients) variables. In the following, assume $E\{X\} = 0$.

Theorem: In the expansion (1), the coefficients c_n become uncorrelated (orthogonal) random variables if and only if $\varphi_n(t)$ are the eigenfunction of the following Fredholm's integral equation:

$$\int_0^T R_{xx}(t_1, t_2)\varphi_n(t_2) = \lambda_n \varphi_n(t_1). \tag{4}$$

In this case,

$$E\{|c_n|^2\} = \lambda_n. \tag{5}$$

Proof: From (1) and (2), it follows that

$$E\{x^*(t_1)c_n\} = E\{|c_n|^2\}\varphi_n = \int_0^T R_{xx}(t_1, t_2)\varphi_n(t_2)dt_2, \tag{6}$$

where

$$E\{c_n c_m^*\} = E\{|c_n|^2\}\delta_{nm} \tag{7}$$

is used. Thus $\lambda_n = E\{|c_n|^2\}$ in equation (4).

It may be also shown that the Karhunen-Loeve ($K - L$) expansion converges in mean-square sense, i.e.

$$E\left\{\left[X(t) - \sum_n c_n \varphi_n(t)\right]^2\right\} = 0. \tag{8}$$

(See Papoulis page 304 for details.) It may also be easily shown that

$$R_{xx}(t_1, t_2) = \sum_n \lambda_n |\varphi_n|^2. \tag{9}$$

Stationary and Periodic Processes

If $X(t)$ is stationary, then $R_{xx} = R_{xx}(t_1 - t_2)$. If in addition, $X(t)$ is also periodic in the mean-square sense, then

$$\varphi_n(t) = \frac{1}{\sqrt{T}} e^{in\omega_0 t}, \qquad \omega_0 = \frac{2\pi}{T}. \tag{10}$$

The $K - L$ expansion for $x(t)$ and R_{xx} are given as

$$x(t) = \sum_{-\infty}^{+\infty} \frac{c_n}{\sqrt{T}} e^{in\omega_o t}, \quad E\{|c_n|^2\} = \lambda_n \tag{11}$$

$$R_{xx}(t_1, t_2) = \frac{1}{T} \sum_{-\infty}^{+\infty} \lambda_n e^{in\omega_0(t_1 - t_2)} \tag{12}$$

The power spectrum of $X(t)$ is then given by

$$S_{xx}(\omega) = \frac{1}{T} \sum_{-\infty}^{+\infty} \lambda_n \delta(\omega - n\omega_0). \tag{13}$$

Furthermore,

$$E\{X^2(t)\} = \frac{1}{T} \sum_{-\infty}^{+\infty} \lambda_n. \tag{14}$$

Stationary Nonperiodic Processes

These may be considered to have infinite period. One may write

$$X(t) = \int_{-\infty}^{+\infty} e^{i\omega t} n(\omega)\sqrt{S(\omega)}\,d\omega, \tag{15}$$

where $n(\omega)$ is a white noise in frequency space with

$$E\{n(\omega_1)n(\omega_2)\} = \delta(\omega_1 - \omega_2) \tag{16}$$

Autocorrelation of $X(t)$ then is given as

$$R_{xx}(t_1, t_2) = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} e^{i\omega_1 t_1 - i\omega_2 t_2}\delta(\omega_1 - \omega_2)\sqrt{s(\omega_1)s(\omega_2)}\,d\omega_1 d\omega_2$$

Hence,

$$R_{xx}(t_1 - t_2) = \int_{-\infty}^{+\infty} e^{-i\omega(t_2 - t_1)} s(\omega)\,d\omega, \tag{17}$$

as is expected.

Responses of a Linear System to White Excitations

Consider a linear system

$$L_t X(t) = n(t), \quad X(0) = X'(0) = ... = 0, \tag{18}$$

with

$$R_{nn}(t_1, t_2) = 2\pi S_0 \delta(t_1 - t_2). \tag{19}$$

Solution to (18) is given by

$$X(t) = \int_0^t h(t-\tau) n(\tau)\,d\tau, \tag{20}$$

where $h(t)$ is the impulse response. i.e.,

$$L_t h(t) = \delta(t). \tag{21}$$

Multiplying (20) by $X(t_2)$ and apply L_t, after averaging one finds

$$L_t R_{xx}(t, t_2) = L_t \int_0^t h(t-\tau) E\{n(\tau) X(t_2)\} d\tau$$

or

$$L_t R_{xx}(t, t_2) = E\{n(t) X(t_2)\} = 2\pi S_0' h(t_2 - t). \tag{22}$$

Restating equation (4) as

$$\int_o^T R_{xx}(t, t_2) \varphi(t_2) dt_2 = \lambda \varphi(t) \tag{23}$$

and applying L_t and using (22), it follows that

$$\int_0^T 2\pi S_0 h(t_2 - t) \varphi(t_2) dt_2 = \lambda L_t \phi(t) \tag{24}$$

Operating on (24) with L_{-t} (same as L_t with t being replaced by $-t$) and using (21), it follows that

$$\lambda L_{-t} L_t \phi(t) = \int_0^T 2\pi S_0 \delta(t_2 - t) \phi(t_2) dt_2 = 2\pi S_0 \varphi(t) \tag{25}$$

Equation (25) is a differential equation for evaluating the eigenfunction $\varphi_n(t)$ and eigenvalues λ_n. It may be shown that if L_t involves derivations of the order N, then the following boundary conditions may be used:

$$\varphi^{(i)}(0) = 0 \qquad \text{for } i = 0,1,\ldots, N-1 \tag{26}$$

$$L_t \varphi^{(i)}(t)|_{t=T} = 0 \qquad \text{for } i = 0,1,\ldots, N-1 \tag{27}$$