ME 639 - Turbulence

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Probability Density Function (pdf) Turbulence Models

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pdf Turbulence Models

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Outline

- Definitions of fine grain pdf and pdf
- Derivation of pdf transport equation from the Navier-Stokes
- Lungren and Chung Models
- Chapman Enskog Procedure and Constitutive Equations
- Model Predictions
- Comparison with Experimental Data

Lundgen's pdf Formulation

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Fine Grain pdf

$$\Im = \delta(\mathbf{u}(\mathbf{x},t) - \mathbf{U})$$

pdf

$$f(\mathbf{U},\mathbf{x},t) = <\delta(\mathbf{u}-\mathrm{U})> = <\Im>$$

Navier-Stokes

$$\frac{\partial \Im}{\partial t} = \frac{\partial \Im}{\partial u_i}\frac{\partial u_i}{\partial t} = -\frac{\partial \Im}{\partial U_i}\frac{\partial u_i}{\partial t}$$

$$\frac{\partial u_i}{\partial t} + u_j \frac{\partial u_i}{\partial x_j} = -\frac{1}{\rho}\frac{\partial P}{\partial x_i} + \nu \frac{\partial^2 u_i}{\partial x_j \partial x_j}$$

$$\frac{\partial u_i}{\partial x_i} = 0$$

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$$\frac{\partial \Im}{\partial t} = -\frac{\partial \Im}{\partial U_i}\left(-u_j \frac{\partial u_i}{\partial x_j} - \frac{1}{\rho}\frac{\partial P}{\partial x_i} + \nu \frac{\partial^2 u_i}{\partial x_j \partial x_j}\right)$$

$$-\frac{\partial \Im}{\partial U_i}\left(-u_j \frac{\partial u_i}{\partial x_j}\right) = \frac{\partial \Im}{\partial u_i}\left(-u_j \frac{\partial u_i}{\partial x_j}\right) = -u_j \frac{\partial \Im}{\partial u_i}\frac{\partial u_i}{\partial x_j} = -u_j \frac{\partial \Im}{\partial x_j} = -\frac{\partial (u_j \Im)}{\partial x_j}$$

$$\frac{\partial \Im}{\partial t} + \frac{\partial}{\partial x_i}(u_i \Im) = \frac{\partial}{\partial U_i}\left(\Im \frac{\partial}{\partial x_i}\left(\frac{P}{\rho}\right)\right) - \nu \frac{\partial}{\partial U_i}\left[\left(\Im \frac{\partial^2 u_i}{\partial x_j \partial x_j}\right)\right]$$

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$$\frac{\partial f}{\partial t} + U_j \frac{\partial f}{\partial x_j} = \frac{\partial}{\partial U_i} < \Im \frac{\partial}{\partial x_i}\left(\frac{P}{\rho}\right) > -\nu \frac{\partial}{\partial U_i} < \Im \frac{\partial^2 u_i}{\partial x_j \partial x_j} >$$

$$\frac{\partial f}{\partial t} + U_j \frac{\partial f}{\partial x_j} = \frac{\partial}{\partial U_i} < \delta(\mathbf{u} - \mathbf{U}) \frac{\partial}{\partial x_i}\left(\frac{P}{\rho}\right) > -\nu \frac{\partial}{\partial U_i} < \delta(\mathbf{u} - \mathbf{U}) \frac{\partial^2 u_i}{\partial x_j \partial x_j} >$$

Second Order pdf

$$f_2(\mathbf{U}_1, \mathbf{U}_2; \mathbf{x}_1, \mathbf{x}_2, t) = < \delta(\mathbf{u}(\mathbf{x}_1, t) - \mathbf{U}_1)\delta(\mathbf{u}(\mathbf{x}_2, t) - \mathbf{U}_2) >$$

Lundgen's pdf Formulation

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$$\frac{\partial f}{\partial t} + U_j \frac{\partial f}{\partial x_j} = \int_{\mathbf{x}'} d\mathbf{x}' \frac{\partial G(\mathbf{x}, \mathbf{x}')}{\partial x_i} \frac{\partial^2}{\partial x'_m \partial x'_n} \frac{\partial}{\partial U_i} \int_{\mathbf{U}'} f_2(\mathbf{U}, \mathbf{U}', \mathbf{x}, \mathbf{x}', t) U'_m U'_n d\mathbf{U}' - \nu \lim_{\mathbf{x}' \to \mathbf{x}} \frac{\partial^2}{\partial x'_j \partial x'_j} \int_{\mathbf{U}'} U'_i \frac{\partial}{\partial U_i} f_2(\mathbf{U}, \mathbf{U}', \mathbf{x}, \mathbf{x}', t) d\mathbf{U}'$$

First Order pdf Closure Methods

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$$\frac{\partial f}{\partial t} + U_j \frac{\partial f}{\partial x_j} + K_i \frac{\partial f}{\partial U_i} = \frac{\partial}{\partial U_i} < \delta(\mathbf{u} - \mathrm{U}) \frac{\partial}{\partial x_i} (\frac{P'}{\rho}) > -\nu \frac{\partial}{\partial U_i} < \delta(\mathbf{u} - \mathbf{U}) \frac{\partial^2 u'_i}{\partial x_j \partial x_j} >$$

$$K_i = \nu \frac{\partial^2 \overline{u_i}}{\partial x_j \partial x_j} - \frac{1}{\rho} \frac{\partial \overline{P}}{\partial x_i}$$

$$u'_i = u_i - \overline{u_i}$$

$$P' = P - \overline{P}$$

Lundgren Relaxation Model

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$$\frac{\partial f}{\partial t} + U_j \frac{\partial f}{\partial x_j} + K_i \frac{\partial f}{\partial U_i} = \beta(f_0 - f) + \beta^\nu \frac{\partial}{\partial U_i}\left[(U_i - \overline{u_i})f\right]$$

$$- < \frac{\partial \Im}{\partial u_i} \frac{\partial}{\partial x_i}\left(\frac{P'}{\rho}\right) > \approx \beta(f_0 - f)$$

$$\beta = \frac{3\kappa}{2k}\left(\varepsilon + \frac{d}{dt}k\right)$$

$$\nu < \Im \nabla^2 u'_i > \approx -\beta^\nu (U_i - \overline{u_i}) f$$

$$\beta^\nu = \frac{2\varepsilon}{k}$$

Chung Model (Fokker-Planck Equation)

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$$-\left\langle \frac{\partial \Im}{\partial u_i}\frac{\partial}{\partial x_i}\left(\frac{P'}{\rho}\right)\right\rangle = \beta\left\{\frac{\partial}{\partial U_i}\left[\left(U_i-\bar{u}_i\right)f\right] + \frac{2k}{3}\frac{\partial^2 f}{\partial U_i \partial U_i}\right\}$$

$$\frac{\partial f}{\partial t} + U_j\frac{\partial f}{\partial x_j} + K_i\frac{\partial f}{\partial U_i} - \beta^{\nu}\frac{\partial}{\partial U_i}\left[\left(U_i-\bar{u}_i\right)f\right] = \beta\left\{\frac{\partial}{\partial U_i}\left[\left(U_i-\bar{u}_i\right)f\right] + \frac{2k}{3}\frac{\partial^2 f}{\partial U_i \partial U_i}\right\}$$

$$\beta = A\frac{k^{\frac{1}{2}}}{\Lambda} \qquad \beta^{\nu} = A'\frac{\nu}{\lambda^2}$$

Chapman-Enskog Approximation

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Solution for pdf

$$f = f_0\left\{1 - \frac{3\nu_T}{2k}\left[\left(\frac{3c^2}{4k} - \frac{5}{2}\right)\frac{c_i}{k}\frac{\partial k}{\partial x_i} + \frac{3c_i d_{ij} c_j}{2k}\right]\right\}$$

$$c_i = U_i - \bar{u}_i$$

$$\nu_T = \frac{4k^2}{9\kappa\left(\varepsilon + \frac{dk}{dt}\right)} = \frac{2}{3}k\tau$$

$$d_{ij} = \frac{1}{2}\left(\bar{u}_{i,j} + \bar{u}_{j,i}\right)$$

Equations of Balance

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Multiplying pdf equation by 1, c and $c^2/2$ and integration over the velocity space yield:

Momentum

$$\frac{d\bar{u}_i}{dt} = -\frac{1}{\rho}\frac{\partial \bar{P}}{\partial x_i} - \frac{\partial}{\partial x_j}P_{ij}$$

Mass

$$\bar{u}_{i,i} = 0$$

$$P_{ij} = \int_c c_i c_j f d\mathbf{c}$$

Energy

$$\frac{dk}{dt} + \frac{\partial Q_i}{\partial x_i} + P_{ij}d_{ij} = -\varepsilon$$

$$Q_i = \frac{1}{2}\int c_i c^2 f d\mathbf{c}$$

Constitutive Equations

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$$P_{ij} = \frac{2}{3}k\delta_{ij} - 2\nu_T d_{ij}$$

$$Q_i = -\kappa_T\frac{\partial}{\partial x_i}k$$

$$2\nu_T^2 d_{ij}d_{ij} + \frac{5}{3}\nu_T\frac{\partial}{\partial x_i}\left(\nu_T\frac{\partial k}{\partial x_i}\right) = \frac{4k^2}{9\kappa}$$

$$\kappa_T = \frac{5}{3}\nu_T$$

When $\frac{dk}{dt} \approx 0$ → $\nu_T = \frac{4k^2}{9\kappa\varepsilon}$ $\qquad \varepsilon \approx \left(\frac{2}{3}\right)^3\frac{k^{\frac{3}{2}}}{\Lambda}$

$$\nu_T = \frac{3k^{\frac{1}{2}}\Lambda}{2\kappa}$$

Couette Flows

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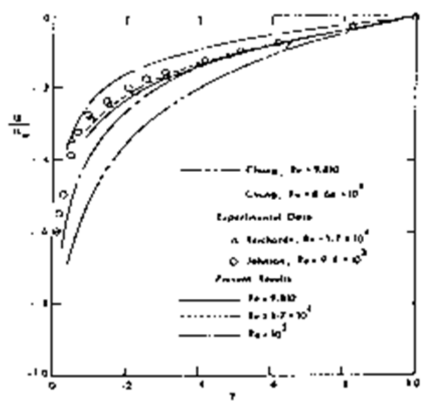

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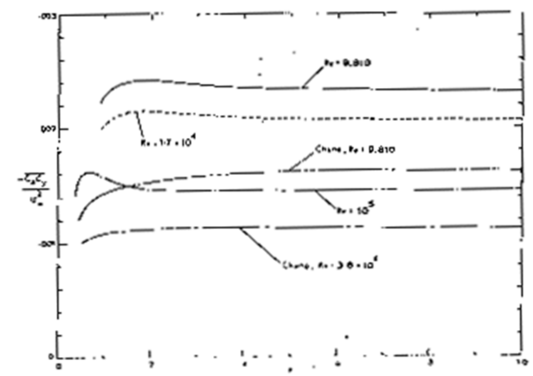

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Conclusions

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- pdf transport equation contains all order moments.
- pdf transport equation can be derived form the Navier-Stokes Eq.
- Lungren and Chung closures models give reasonable results for simple flows.

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Thank you!

Questions?

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