Probability Density Formulation (pdf) in Turbulence

Lundgen's pdf Formulation [Phys. Fluids (1967) Vol. 10, 969.]

Let

$$\Im = \delta(\mathbf{u}(\mathbf{x}, t) - \mathbf{U}) \tag{1}$$

then the probability density function of $\mathbf{u}$ being $\mathbf{U}$ is given by

$$f(\mathbf{U}, \mathbf{x}, t) = < \delta(\mathbf{u} - U) > = < \Im > . \tag{2}$$

Now evaluating the time derivative of $\Im$ and using chain rule, we find

$$\frac{\partial \Im}{\partial t} = \frac{\partial \Im}{\partial u_i}\frac{\partial u_i}{\partial t} = -\frac{\partial \Im}{\partial U_i}\frac{\partial u_i}{\partial t} \tag{3}$$

But u_i satisfies the Navier-Stokes equation and continuity. These are

$$\frac{\partial u_i}{\partial t} + u_j \frac{\partial u_i}{\partial x_j} = -\frac{1}{\rho}\frac{\partial P}{\partial x_i} + \nu \frac{\partial^2 u_i}{\partial x_j \partial x_j}, \qquad \frac{\partial u_i}{\partial x_i} = 0 \tag{4}$$

Using Equation (4) in (3) we find

$$\frac{\partial \Im}{\partial t} = -\frac{\partial \Im}{\partial U_i}\left(-u_j \frac{\partial u_i}{\partial x_j} - \frac{1}{\rho}\frac{\partial P}{\partial x_i} + \nu \frac{\partial^2 u_i}{\partial x_j \partial x_j} \right) \tag{5}$$

Note that

$$-\frac{\partial \Im}{\partial U_i}\left(-u_j \frac{\partial u_i}{\partial x_j} \right) = \frac{\partial \Im}{\partial u_i}\left(-u_j \frac{\partial u_i}{\partial x_j} \right) = -u_j \frac{\partial \Im}{\partial u_i}\frac{\partial u_i}{\partial x_j} = -u_j \frac{\partial \Im}{\partial x_j} = -\frac{\partial (u_j \Im)}{\partial x_j} \tag{6}$$

Equation (5) may be restated as

$$\frac{\partial \Im}{\partial t} + \frac{\partial}{\partial x_i}(u_i \Im) = \frac{\partial}{\partial U_i}\left(\Im \frac{\partial}{\partial x_i}\left(\frac{P}{\rho} \right) \right) - \nu \frac{\partial}{\partial U_i}\left[\left(\Im \frac{\partial^2 u_i}{\partial x_j \partial x_j} \right) \right] \tag{7}$$

Taking expected value of (7), it follows that

$$\frac{\partial f}{\partial t} + U_j \frac{\partial f}{\partial x_j} = \frac{\partial}{\partial U_i} < \Im \frac{\partial}{\partial x_i}\left(\frac{P}{\rho}\right) > -\nu \frac{\partial}{\partial U_i} < \Im \frac{\partial^2 u_i}{\partial x_j \partial x_j} > \tag{8}$$

or

$$\frac{\partial f}{\partial t} + U_j \frac{\partial f}{\partial x_j} = \frac{\partial}{\partial U_i} < \delta(\mathbf{u} - \mathbf{U}) \frac{\partial}{\partial x_i}\left(\frac{P}{\rho}\right) > -\nu \frac{\partial}{\partial U_i} < \delta(\mathbf{u} - \mathbf{U}) \frac{\partial^2 u_i}{\partial x_j \partial x_j} > \tag{9}$$

Equation (9) governs the probability density function of turbulent flow. It is interesting to note that the nonlinear convective term is automatically taken care of, but the pressure and viscous terms need to be closed.

Lundgren introduced the second order joint density defined as

$$f_2(\mathbf{U_1}, \mathbf{U_2}; \mathbf{x_1}, \mathbf{x_2}, t) = < \delta(\mathbf{u}(\mathbf{x_1}, t) - \mathbf{U_1}) \delta(\mathbf{u}(\mathbf{x_2}, t) - \mathbf{U_2}) > \tag{10}$$

Also taking divergence of Navier-Stokes equation, and solving the result for pressure, it follows that

$$\frac{P(\mathbf{x})}{\rho} = \int_{\mathbf{x}'} G(\mathbf{x}, \mathbf{x}') \frac{\partial u_i'}{\partial x_j'} \frac{\partial u_j'}{\partial x_i'} d\mathbf{x}' \tag{11}$$

where $G(\mathbf{x}, \mathbf{x}')$ is the Green function for ∇^2 operator. The pressure term in the pdf equation may be restated as

$$\begin{aligned} < \delta(\mathbf{u} - \mathbf{U}) \frac{\partial}{\partial x_i}\left(\frac{P}{\rho}\right) > &= \int_{\mathbf{x}'} \frac{\partial G(\mathbf{x}, \mathbf{x}')}{\partial x_i} < \delta(\mathbf{u}(x) - \mathbf{U}) \frac{\partial u_m'}{\partial x_n'} \frac{\partial u_n'}{\partial x_m'} > d\mathbf{x}' \\ &= \int_{x'} \frac{\partial G(\mathbf{x}, \mathbf{x}')}{\partial x_i} \frac{\partial^2}{\partial x_m' \partial x_n'} < \delta[\mathbf{u}(x) - \mathbf{U}] u_m' u_n' > d\mathbf{x}' \\ &= \int_{\mathbf{x}'} \frac{\partial G(\mathbf{x}, \mathbf{x}')}{\partial x_i} \frac{\partial^2}{\partial x_m' \partial x_n'} \int_{\mathbf{U}_1'} f_2(\mathbf{U}, \mathbf{U}', \mathbf{x}, \mathbf{x}', t) U_m' U_n' d\mathbf{U}' d\mathbf{x} \end{aligned} \tag{12}$$

Similarly, the viscous term can be written as

$$\begin{aligned} < \delta(\mathbf{u} - \mathbf{U}) \frac{\partial^2 u_i}{\partial x_j \partial x_j} > &= < \delta(\mathbf{u}(\mathbf{x}) - \mathbf{U}) \int_{\mathbf{U}'} \lim_{\mathbf{x}' \to \mathbf{x}} \frac{\partial^2}{\partial x_j' \partial x_j'} [\delta(\mathbf{u}(\mathbf{x}') - \mathbf{U}') U_i'] d\mathbf{U}' > \\ &= \lim_{\mathbf{x}' \to \mathbf{x}} \frac{\partial^2}{\partial x_j' \partial x_j'} \int_{\mathbf{U}'} U_i' f_2(\mathbf{U}, \mathbf{U}', \mathbf{x}, \mathbf{x}', t) d\mathbf{U}' \end{aligned} \tag{13}$$

Therefore,

$$\frac{\partial f}{\partial t}+U_j\frac{\partial f}{\partial x_j}=\int_{\mathbf{x}'}d\mathbf{x}'\frac{\partial G(\mathbf{x},\mathbf{x}')}{\partial x_i}\frac{\partial^2}{\partial x'_m\partial x'_n}\frac{\partial}{\partial U_i}\int_{\mathbf{U}'}f_2(\mathbf{U},\mathbf{U}',\mathbf{x},\mathbf{x}',t)U'_mU'_nd\mathbf{U}'$$
$$-\nu\lim_{\mathbf{x}'\to\mathbf{x}}\frac{\partial^2}{\partial x'_j\partial x'_j}\int_{\mathbf{U}'}U'_i\frac{\partial}{\partial U_i}f_2(\mathbf{U},\mathbf{U}',\mathbf{x},\mathbf{x}',t)d\mathbf{U}' \quad (14)$$

Equation (14) for the first order pdf involves second order pdf, which is the characteristics difficulty of the turbulence closed problem.

Approximate First Order pdf Closure Methods

The transport equation for the first order pdf as given Equation (9) may be restated as

$$\frac{\partial f}{\partial t}+U_j\frac{\partial f}{\partial x_j}+K_i\frac{\partial f}{\partial U_i}=\frac{\partial}{\partial U_i}<\delta(\mathbf{u}-U)\frac{\partial}{\partial x_i}(\frac{P'}{\rho})>-\nu\frac{\partial}{\partial U_i}<\delta(\mathbf{u}-\mathbf{U})\frac{\partial^2 u'_i}{\partial x_j\partial x_j}> \quad (15)$$

where

$$K_i=\nu\frac{\partial^2\overline{u_i}}{\partial x_j\partial x_j}-\frac{1}{\rho}\frac{\partial\overline{P}}{\partial x_i}, \qquad u'_i=u_i-\overline{u_i}, \quad P'=P-\overline{P}. \quad (16)$$

Lundgren Relaxation Model

Lungren obtained the following relaxation model for first order pdf :

$$\frac{\partial f}{\partial t}+U_j\frac{\partial f}{\partial x_j}+K_i\frac{\partial f}{\partial U_i}=\beta(f_0-f)+\beta^\nu\frac{\partial}{\partial U_i}\left[(U_i-\overline{u_i})f\right] \quad (17)$$

In deriving Equation (17) the following closure assumptions are used:

$$-<\frac{\partial\Im}{\partial u_i}\frac{\partial}{\partial x_i}\left(\frac{P'}{\rho}\right)>\approx\beta(f_0-f) \quad (18)$$

$$\nu<\Im\nabla^2u'_i>\approx-\beta^\nu(U_i-\overline{u_i})f \quad (19)$$

where

$$\beta = \frac{3\kappa}{2k}\left(\varepsilon + \frac{d}{dt}k\right) \tag{20}$$

$$\beta^{\nu} = \frac{2\varepsilon}{k} \tag{21}$$

Here, $\kappa = 4$, $k = \frac{1}{2}\overline{u_i' u_i'}$, and ε is the dissipation rate.

Chung Model (Fokker-Planck Equation)

An alternative Fokker-Planck type closure model was suggested by Chung. That is,

$$-\left\langle \frac{\partial \mathfrak{I}}{\partial u_i}\frac{\partial}{\partial x_i}\left(\frac{P'}{\rho}\right)\right\rangle = \beta\left\{\frac{\partial}{\partial U_i}\left[\left(U_i - \overline{u_i}\right)f\right] + \frac{2k}{3}\frac{\partial^2 f}{\partial U_i \partial U_i}\right\} \tag{22}$$

The corresponding closed pdf equation becomes

$$\frac{\partial f}{\partial t} + U_j \frac{\partial f}{\partial x_j} + K_i \frac{\partial f}{\partial U_i} - \beta^{\nu}\frac{\partial}{\partial U_i}\left[\left(U_i - \overline{u_i}\right)f\right] = \beta\left\{\frac{\partial}{\partial U_i}\left[\left(U_i - \overline{u_i}\right)f\right] + \frac{2k}{3}\frac{\partial^2 f}{\partial U_i \partial U_i}\right\} \tag{23}$$

where

$$\beta = A\frac{k^{\frac{1}{2}}}{\Lambda}, \qquad \beta^{\nu} = A'\frac{\nu}{\lambda^2} \tag{24}$$

Chung original derivation follows from a Langevin equation.

Chapman-Enskog Approximation

Using the Chapman-Enskog approximation method, Lundgren (Phys. Fluids, Vol. 12, 485 (1969)) found a solution for the first order pdf. That is

$$f = f_0\left\{1 - \frac{3\nu_T}{2k}\left[\left(\frac{3c^2}{4k} - \frac{5}{2}\right)\frac{c_i}{k}\frac{\partial k}{\partial x_i} + \frac{3c_i d_{ij} c_j}{2k}\right]\right\}, \tag{25}$$

where

$$c_i = U_i - \overline{u_i}, \quad d_{ij} = \frac{1}{2}(\bar{u}_{i,j} + \bar{u}_{j,i}) \tag{26}$$

and

$$\nu_T = \frac{4k^2}{9\kappa\left(\varepsilon + \frac{dk}{dt}\right)} = \frac{2}{3}k\tau \tag{27}$$

Here t is the relaxation time and k is a constant

Equations of Balance

Multiplying the pdf transport equation by 1, c, and $\frac{1}{2}c^2$ and integrating over the entire velocity space, we find

$$\bar{u}_{i,i} = 0\,, \tag{28}$$

$$\frac{d\overline{u_i}}{dt} = -\frac{1}{\rho}\frac{\partial \bar{P}}{\partial x_i} - \frac{\partial}{\partial x_j}P_{ij}\,, \tag{29}$$

$$\frac{dk}{dt} + \frac{\partial Q_i}{\partial x_i} + P_{ij}d_{ij} = -\varepsilon\,, \tag{30}$$

where

$$P_{ij} = \int_c c_i c_j f d\mathbf{c}\,, \tag{31}$$

$$Q_i = \frac{1}{2}\int c_i c^2 f d\mathbf{c}\,. \tag{32}$$

Note that the viscous diffusion term is neglected.

Constitutive Equations

The constitutive equations become

$$P_{ij} = \frac{2}{3}k\delta_{ij} - 2\nu_T d_{ij} \tag{33}$$

$$Q_i = -\kappa_T \frac{\partial}{\partial x_i}k\,, \qquad \kappa_T = \frac{5}{3}\nu_T \tag{34}$$

the eddy viscosity satisfies the following equation

$$2\nu_T^2 d_{ij} d_{ij} + \frac{5}{3}\nu_T \frac{\partial}{\partial x_i}\left(\nu_T \frac{\partial k}{\partial x_i}\right) = \frac{4k^2}{9\kappa} \qquad (35)$$

Limiting cases for ν_T *:*

When diffusion is small,

$$\nu_T^2 = \frac{2k^2}{9\kappa d_{ij} d_{ij}} \qquad (36)$$

When $\frac{dk}{dt} \approx 0$,

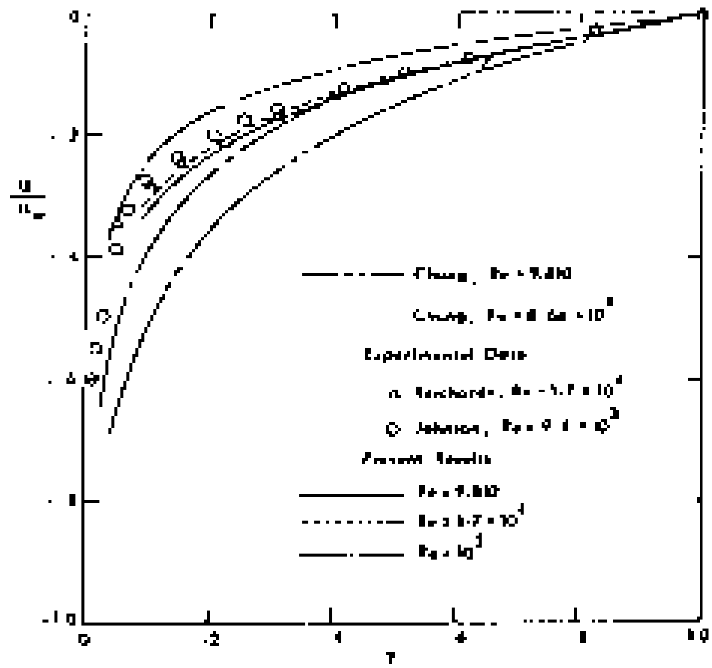
$$\nu_T = \frac{4k^2}{9\kappa\varepsilon}, \qquad \varepsilon \approx \left(\frac{2}{3}\right)^3 \frac{k^{\frac{3}{2}}}{\Lambda} \qquad (37)$$

Therefore,

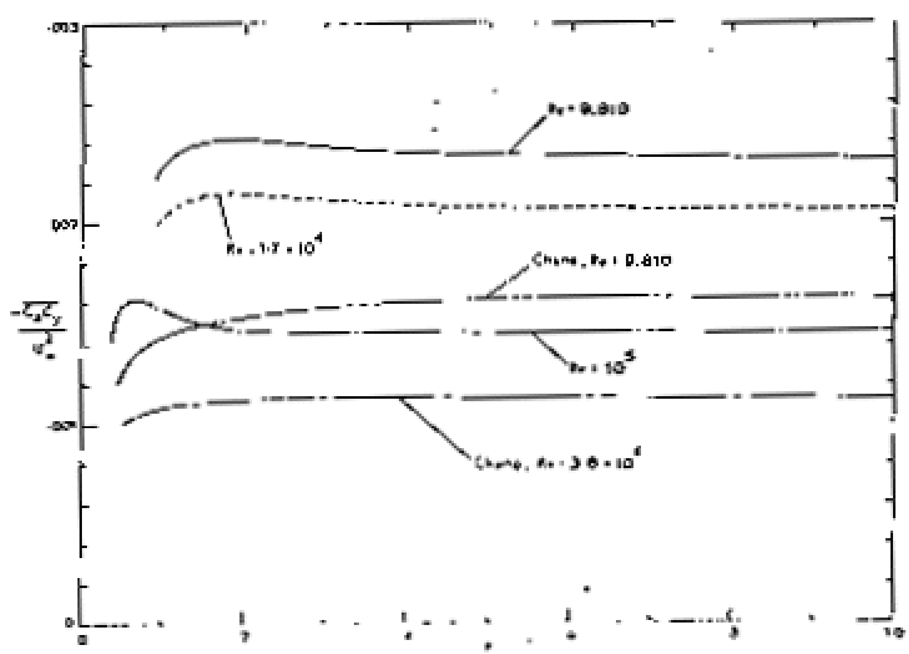
$$\nu_T = \frac{3k^{\frac{1}{2}}\Lambda}{2\kappa} \qquad (35)$$

Chapman-Enskog method gives qualitatively reasonable results (with about 10 to 30% error) for simple shear flows. Numerical results for the case of a turbulent Couette flow obtained by Srinivansan et al. (Phys Fluids, Vol. 20, 554 (1977)) is described in following section.

Couette Flow



Mean flow velocity profiles.



Reynolds stress profiles.