ME 639 - Turbulence

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Turbulence Modeling

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Turbulence

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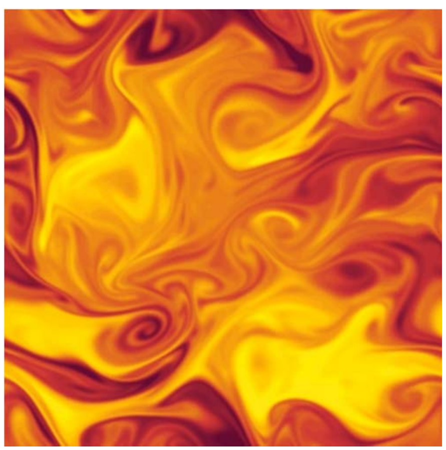

Turbulence Modeling

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Outline

- Viscous Flow
- Turbulence
- Mixing Length Models
- One-Equation Models
- Two-Equation Models
- Stress Transport Models
- Rate-Dependent Models

Viscous Flows

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Mass → $$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = 0$$

Momentum $$\rho \frac{d\mathbf{u}}{dt} = \rho \mathbf{f} + \nabla \cdot \boldsymbol{\tau}$$ $$\boldsymbol{\tau}^T = \boldsymbol{\tau}$$

Energy → $$\rho \dot{e} = \boldsymbol{\tau} : \nabla \mathbf{u} + \nabla \mathbf{q} + \rho h$$

Entropy → $$\rho \dot{\eta} - \nabla \cdot \left(\frac{\mathbf{q}}{T}\right) - \frac{\rho h}{T} \geq 0$$

Viscous Flows Constitutive Equations

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Viscous Fluids → $$\tau_{kl} = -p\delta_{kl} + G_{kl}(u_{i,j})$$

$$u_{i,j} = d_{ij} + \omega_{ij}$$

$$d_{kl} = \frac{1}{2}(u_{k,l} + u_{l,k})$$

$$\omega_{kl} = \frac{1}{2}(u_{k,l} - u_{l,k})$$

Material Frame-Indifference → $$\tau_{kl} = -p\delta_{kl} + F_{kl}(d_{ij})$$

Constitutive Equations Newtonian Fluids

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$$\tau_{kl} = (-p + \lambda u_{i,i})\delta_{kl} + 2\mu d_{kl}$$

$$\mu \geq 0$$

$$3\lambda + 2\mu \geq 0$$

Navier- Stokes

$$\rho\left(\frac{\partial u_i}{\partial t} + u_j \frac{\partial u_i}{\partial x_j}\right) = -\frac{\partial p}{\partial x_i} + \mu \frac{\partial^2 u_i}{\partial x_j \partial x_j}$$

$$\frac{\partial u_i}{\partial x_i} = 0$$

Turbulent Flows

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$$u_i = U_i + u_i'$$ $$\overline{u_i} = U_i$$ $$\overline{u_i'} = 0$$

$$p = P + p'$$ $$\overline{p} = P$$ $$\overline{p'} = 0$$

Reynolds Equation

$$\frac{\partial U_i}{\partial t} + U_j \frac{\partial U_i}{\partial x_j} = -\frac{1}{\rho}\frac{\partial P}{\partial x_i} + \nu \frac{\partial^2 U_i}{\partial x_j x_j} - \frac{\partial \overline{u_i' u_j'}}{\partial x_j}$$

Turbulent Stress → $$\tau_{ij}^T = -\rho \overline{u_i' u_j'}$$

First Order Modeling

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Eddy Viscosity

$$\tau_{21}^T = -\rho \overline{u'v'} = \rho \nu_T \frac{dU}{dy}$$

$$\frac{\tau_{ij}^T}{\rho} = -\overline{u_i' u_j'} = \nu_T \left(\frac{\partial U_i}{\partial x_j} + \frac{\partial U_j}{\partial x_i}\right) - \frac{1}{3}\overline{u_k' u_k'}\,\delta_{ij}$$

Mixing Length

$$\tau_{21}^T = \rho l_m^2 \left|\frac{\partial U}{\partial y}\right| \frac{\partial U}{\partial y} \qquad -\overline{T'v'} = \frac{\nu_T}{\sigma_T}\frac{\partial T}{\partial y}$$

$$\nu_T = l_m^2 \left|\frac{\partial U}{\partial y}\right|$$

Kolmogorov-Prandtl Expression

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Eddy Viscosity → $\nu_T \approx c\,u\,\ell$; $u =$ Velocity Scale ; $\ell =$ Length Scale

Kinematic Viscosity → $\nu \propto c\lambda$; $c =$ Speed of Sound ; $\lambda =$ Mean Free Path

Free Shear Flows → $\ell_m \simeq c\ell_0$

Near Wall Flows → $\ell_m = \kappa y$

Evaluation of Constants Inertial Sublayer

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Local Equilibrium Production = Dissipation ↔ **Mixing length Hypothesis**

Short Comings of Mixing Length

- Eddy viscosity vanishes when velocity gradient is zero
- Lack of transport of turbulence scales
- Estimating the mixing length

Shortcomings of Mixing Length Hypothesis

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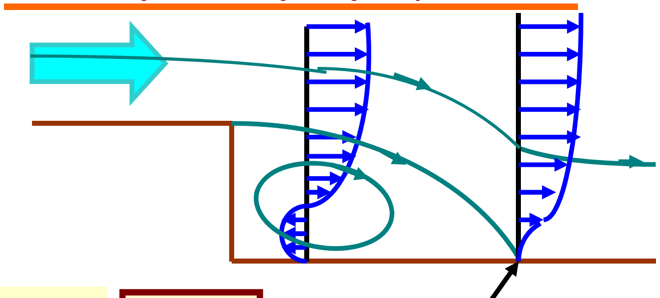

Reattachment Point

Mixing Length : $\gamma_T = 0$

Experiment → **Maximum Heat Transfer**

One-Equation Models

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Eddy Viscosity → $\nu_T = c_\mu k^{1/2} \ell$

Exact k-equation

$$\underbrace{\frac{d}{dt}\frac{\overline{u_i'u_i'}}{2}}_{\text{Convective Transport}} = \underbrace{-\frac{\partial}{\partial x_k}\overline{u_k'\left(\frac{u_i'u_i'}{2}+\frac{P'}{\rho}\right)}}_{\text{Turbulence Diffusion}} \underbrace{-\overline{u_i'u_j'}\frac{\partial U_i}{\partial x_j}}_{\text{Production}}$$

$$\underbrace{-\nu\overline{\frac{\partial u_i'}{\partial x_j}\frac{\partial u_i'}{\partial x_j}}}_{\text{Dissipation}} + \underbrace{\nu\frac{\partial^2}{\partial x_j x_j}\frac{\overline{u_i'u_i'}}{2}}_{\text{Viscous Diffusion}}$$

One-Equation Models

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Modeled k-equation

$$\frac{dk}{dt} = \underbrace{\frac{\partial}{\partial x_j}\left(\frac{\nu_T}{\sigma_k}\frac{\partial k}{\partial x_j}\right)}_{\text{Diffusion}}$$

$$+\underbrace{\nu_T\left(\frac{\partial U_i}{\partial x_j}+\frac{\partial U_j}{\partial x_i}\right)\frac{\partial U_i}{\partial x_j}}_{\text{Production}} \underbrace{- c_D\frac{k^{3/2}}{\ell}}_{\text{Dissipation}}$$

Bradshaw's Model

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K-equation

$$\frac{dk}{dt} = \underbrace{\frac{\partial}{\partial y}\left(Bk\sqrt{\frac{\tau_{Max}}{\rho}}\right)}_{\text{Diffusion}} + \underbrace{ak\frac{\partial U}{\partial y}}_{\text{Production}} \underbrace{- c_D\frac{k^{3/2}}{\ell}}_{\text{Dissipation}}$$

Short Comings of One-Equation Models

- Lack of transport of turbulence length scale
- Estimating the length scale

z-Equation

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$$\frac{dz}{dt} = \underbrace{\frac{\partial}{\partial y}\left(\frac{\nu_T}{\sigma_z}\frac{\partial z}{\partial y}\right)}_{\text{Diffusion}} + z[\underbrace{c_1\frac{\nu_T}{k}\left(\frac{\partial U}{\partial y}\right)^2}_{\text{Production}}$$

$$\underbrace{- c_2\frac{k}{\nu_T}}_{\text{Dissipation}}] + \underbrace{S_z}_{\text{Secondary Source}}$$

Choices for z-Scale

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$$\sqrt{\ell^2 / k} = \text{Time Scale}$$

$$z = k\ell$$

$$\sqrt{k / \ell^2} = \text{Frequency Scale}$$

$$k / \ell^2 = \text{Vorticity Scale}$$

$$\varepsilon = \nu \overline{\frac{\partial u'_i}{\partial x_j}\frac{\partial u'_i}{\partial x_j}} = \text{Dissipation Rate}$$

Exact Dissipation Equation

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$$\frac{d\varepsilon}{dt} = \underbrace{-\frac{\partial}{\partial x_j}(\overline{u'_j \varepsilon'})}_{\text{Diffusion}} \underbrace{- 2\nu \overline{\frac{\partial u'_i}{\partial x_k}\frac{\partial u'_i}{\partial x_l}\frac{\partial u'_k}{\partial x_l}}}_{\text{Generation by vortex stretching}}$$

$$\underbrace{- 2\nu \overline{\frac{\partial^2 u'_i}{\partial x_k \partial x_l}\frac{\partial^2 u'_i}{\partial x_k \partial x_l}}}_{\text{Viscous distruction}}$$

Energy Spectrum of Turbulence

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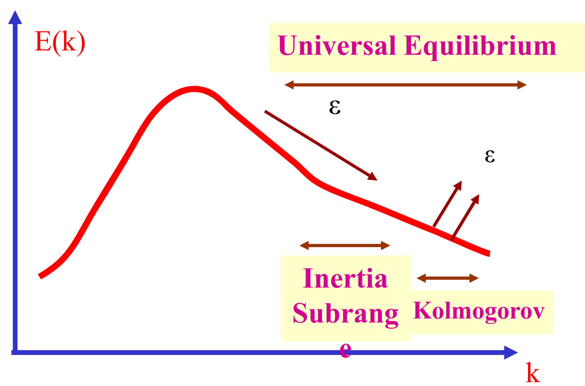

k-e Model

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Mass → $$\frac{\partial U_i}{\partial x_i} = 0$$

Momentum $$\frac{dU_i}{dt} = -\frac{1}{\rho}\frac{\partial P}{\partial x_i} - \frac{\partial}{\partial x_j}\overline{u'_i u'_j}$$

$$\nu_T = \frac{c_\mu k^2}{\varepsilon}$$

$$-\overline{u'_i u'_j} = \nu_T\left(\frac{\partial U_i}{\partial x_j} + \frac{\partial U_j}{\partial x_i}\right) - \frac{2}{3}k\delta_{ij}$$

k-e Model

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k-equation

$$\frac{dk}{dt} = \underbrace{\frac{\partial}{\partial x_j}\left(\frac{\upsilon_T}{\sigma_k}\frac{\partial k}{\partial x_j}\right)}_{\text{Diffusion}} + \underbrace{\nu_T\left(\frac{\partial U_i}{\partial x_j} + \frac{\partial U_j}{\partial x_i}\right)\frac{\partial U_i}{\partial x_j}}_{\text{Pr oduction}} - \underbrace{\varepsilon}_{\text{dissipation}}$$

ε-equation

$$\frac{d\varepsilon}{dt} = \underbrace{\frac{\partial}{\partial x_j}\left(\frac{\upsilon_T}{\sigma_\varepsilon}\frac{\partial \varepsilon}{\partial x_j}\right)}_{\text{Diffusion}} + \underbrace{c_{\varepsilon 1}\nu_T\frac{\varepsilon}{k}\left(\frac{\partial U_i}{\partial x_j} + \frac{\partial U_j}{\partial x_i}\right)\frac{\partial U_i}{\partial x_j}}_{\text{Generation}} - \underbrace{c_{\varepsilon 2}\frac{\varepsilon^2}{k}}_{\text{Distruction}}$$

Development of Plane Mixing Layer (Rodi, 1982)

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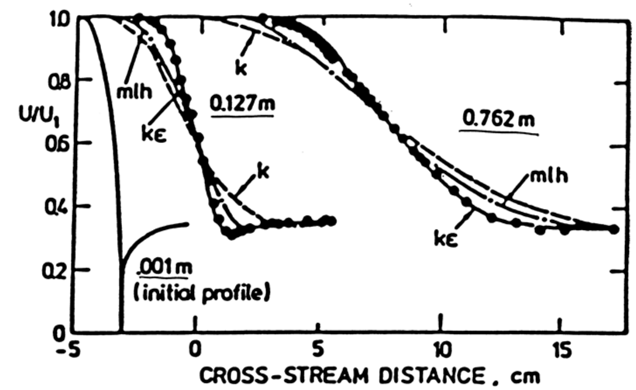

Turbulent Recirculating Flow (Durst and Rastogi, 1979)

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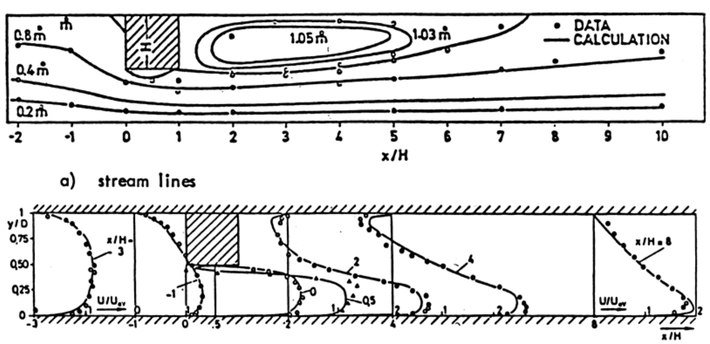

k-ε Model

Flow in a Square Cavity (Gosman and Young)

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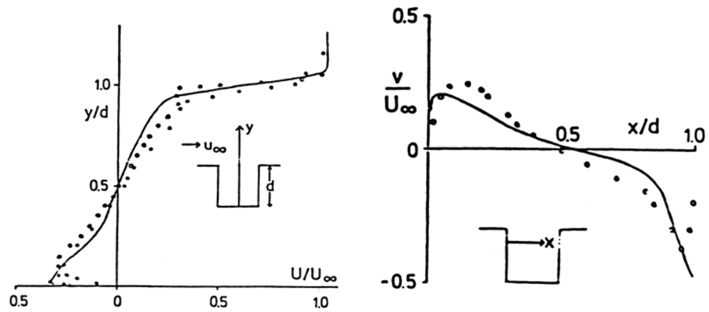

k-ε Model

Free-Stream Turbulence

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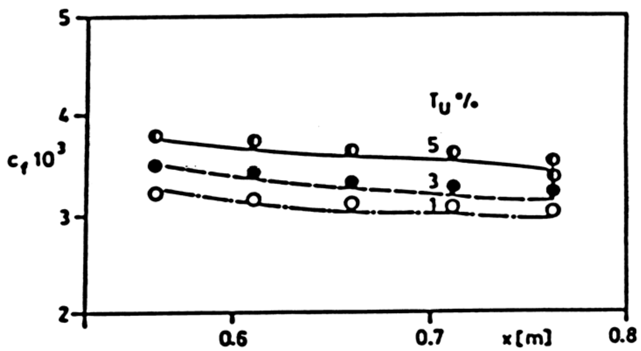

k-ε Model

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Turbulent Channel Flow (Rodi, 1980)

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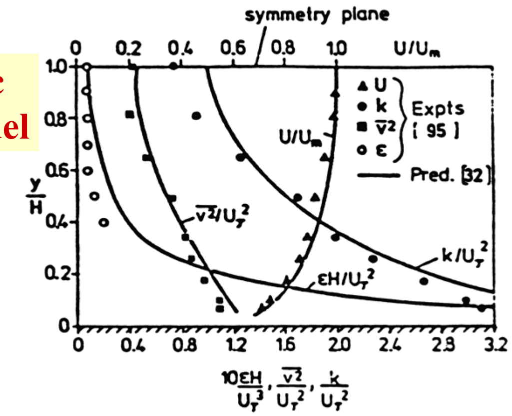
Algebraic Stress Model



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Turbulent Channel Flow (Rodi, 1980)

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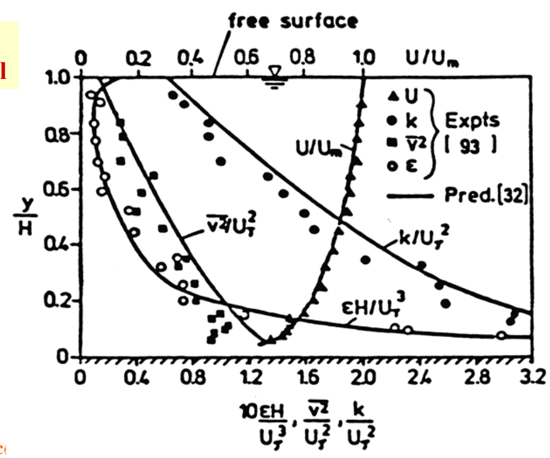
Algebraic Stress Model



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Jets Issuing in Co-flowing Streams (Rodi, 1982)

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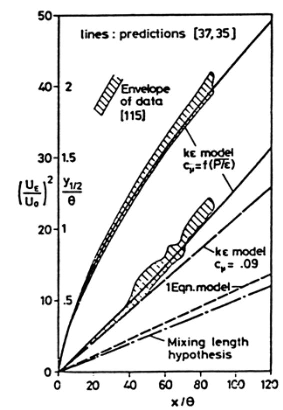

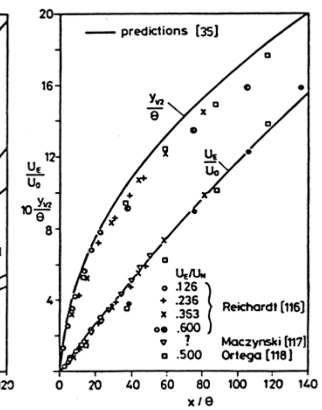

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Short Comings of the k-e Models

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- Eddy viscosity assumption
- Isotropic eddy viscosity
- Negligible convection and diffusion of turbulent shear stress $\overline{u_i' u_j'} \sim k$
- Absence of normal stress effects

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Stress Transport Models

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Fluctuation Velocity

$$\frac{\partial u_i'}{\partial t} + U_k \frac{\partial u_i'}{\partial x_k} = -\frac{1}{\rho}\frac{\partial p'}{\partial x_i} + \nu \frac{\partial^2 u_i'}{\partial x_k \partial x_k} + \frac{\partial}{\partial x_k}\overline{u_i' u_k'} - \frac{\partial}{\partial x_k}(u_i' u_k') - u_k' \frac{\partial U_i}{\partial x_k}$$

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Stress Transport Models

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$$\underbrace{\left(\frac{\partial}{\partial t} + U_k \frac{\partial}{\partial x_k}\right)\overline{u_i' u_j'}}_{\text{Convection}} = \underbrace{-\left[\overline{u_i' u_k'}\frac{\partial U_j}{\partial x_k} + \overline{u_j' u_k'}\frac{\partial U_i}{\partial x_k}\right]}_{\text{Production}} - \underbrace{2\nu\overline{\frac{\partial u_i'}{\partial x_k}\frac{\partial u_j'}{\partial x_k}}}_{\text{Dissipation}} + \underbrace{\overline{\frac{p'}{\rho}\left(\frac{\partial u_i'}{\partial x_j} + \frac{\partial u_j'}{\partial x_i}\right)}}_{\text{Pressure-strain}} \underbrace{- \frac{\partial}{\partial x_k}\left[\overline{u_i' u_j' u_k'} + \overline{\frac{p'}{\rho}(u_i'\delta_{jk} + u_j'\delta_{ik})} - \nu\frac{\partial}{\partial x_k}\overline{u_i' u_j'}\right]}_{\text{Diffusion}}$$

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Stress Transport Models

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Diffusion

$$-\overline{u_i' u_j' u_k'} = c_s \frac{k}{\varepsilon}\left(\overline{u_i' u_l'}\frac{\partial \overline{u_j' u_k'}}{\partial x_l} + \overline{u_j' u_l'}\frac{\partial \overline{u_k' u_i'}}{\partial x_l} + \overline{u_k' u_l'}\frac{\partial \overline{u_i' u_j'}}{\partial x_l}\right)$$

Dissipation

$$2\nu\overline{\frac{\partial u_i'}{\partial x_k}\frac{\partial u_j'}{\partial x_k}} = \frac{2}{3}\delta_{ij}\varepsilon$$

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Stress Transport Models
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Pressure-Strain

$$\overline{\frac{p'}{\rho}\left(\frac{\partial u_i'}{\partial x_j}+\frac{\partial u_j'}{\partial x_i}\right)} = -\int_{\mathbf{x}_1} d\mathbf{x}_1 \, G(\mathbf{x},\mathbf{x}_1)\Big\{ \underbrace{\overline{\left(\frac{\partial u_l'}{\partial x_m}\frac{\partial u_m'}{\partial x_l}\right)_1\left(\frac{\partial u_i'}{\partial x_j}+\frac{\partial u_j'}{\partial x_i}\right)}}_{\varphi_{ij}^{(1)}=\text{Return to Isotropy}} + \underbrace{2\left(\frac{\partial U_l}{\partial x_m}\right)_1 \overline{\left(\frac{\partial u_m'}{\partial x_l}\right)_1\left(\frac{\partial u_i'}{\partial x_j}+\frac{\partial u_j'}{\partial x_i}\right)}}_{\varphi_{ij}^{(2)}+\varphi_{ji}^{(2)}=\text{Rapid Term}} \Big\}$$

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Return to Isotropy

$$\varphi_{ij}^{(1)} = -c_1\left(\frac{\varepsilon}{k}\right)\left(\overline{u_i'u_j'}-\frac{2}{3}\delta_{ij}k\right)$$

Rapid Term

$$\varphi_{ij}^{(2)} + \varphi_{ji}^{(2)} = -\gamma\left(P_{ij}-\frac{2}{3}P\delta_{ij}\right)$$

Production

$$P_{ij} = -\left(\overline{u_i'u_k'}\frac{\partial U_j}{\partial x_k}+\overline{u_j'u_k'}\frac{\partial U_i}{\partial x_k}\right)$$

Stress Transport Models
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$$\underbrace{\left(\frac{\partial}{\partial t}+U_k\frac{\partial}{\partial x_k}\right)\overline{u_i'u_j'}}_{\text{Convection}} = \underbrace{-\left[\overline{u_j'u_k'}\frac{\partial U_i}{\partial x_k}+\overline{u_i'u_k'}\frac{\partial U_j}{\partial x_k}\right]}_{\text{Production}} - \underbrace{\frac{2}{3}\delta_{ij}\varepsilon}_{\text{Dissipation}}$$
$$\underbrace{-c_1\frac{\varepsilon}{k}\left(\overline{u_i'u_j'}-\frac{2}{3}\delta_{ij}k\right)+\left(\varphi_{ij}^{(2)}+\varphi_{ji}^{(2)}\right)}_{\text{Pressure-strain}} + \underbrace{\left(\varphi_{ij}^{(w)}+\varphi_{ji}^{(w)}\right)}_{\text{Wall effects}}$$
$$\underbrace{+c_s\frac{\partial}{\partial x_k}\left\{\frac{k}{\varepsilon}\left[\overline{u_i'u_l'}\frac{\partial \overline{u_j'u_k'}}{\partial x_l}+\overline{u_j'u_l'}\frac{\partial \overline{u_k'u_i'}}{\partial x_l}+\overline{u_k'u_l'}\frac{\partial \overline{u_i'u_j'}}{\partial x_l}\right]\right\}}_{\text{Diffusion}}$$

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Dissipation

$$\underbrace{\left(\frac{\partial}{\partial t}+U_k\frac{\partial}{\partial x_k}\right)\varepsilon}_{\text{Convection}} = \underbrace{c_\varepsilon\frac{\partial}{\partial x_k}\left(\frac{k}{\varepsilon}\overline{u_k'u_i'}\frac{\partial\varepsilon}{\partial x_i}\right)}_{\text{Diffusion}}$$
$$\underbrace{-c_{\varepsilon 1}\frac{\varepsilon}{k}\overline{u_i'u_k'}\frac{\partial U_i}{\partial x_k}}_{\text{Generation}} - \underbrace{c_{\varepsilon 2}\frac{\varepsilon^2}{k}}_{\text{Destruction}}$$

Stress Transport Models

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Reynolds

$$\left(\frac{\partial}{\partial t} + U_j \frac{\partial}{\partial x_j}\right) U_i = -\frac{1}{\rho}\frac{\partial P}{\partial x_i} - \frac{\partial}{\partial x_j}\overline{u_i' u_j'}$$

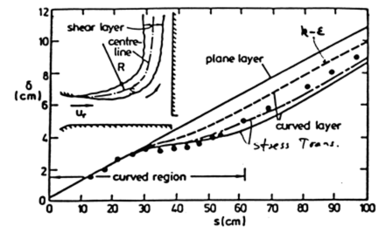
Mass

$$\frac{\partial U_i}{\partial x_i} = 0$$

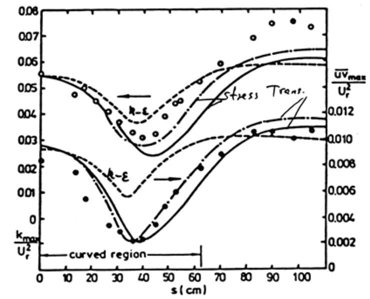
11 Unknowns for 11 Equations

$$U_i,\ \overline{u_i' u_j'},\ P,\ \varepsilon$$

Curved Mixing Layer



Gibson and Rodi (1981)



Asymmetric Channel Flow (Launder, Reece and Rodi, 1975)

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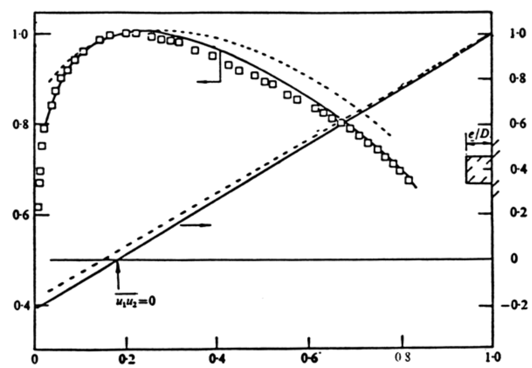

Mean Velocity and Turbulence Shear Stress

Asymmetric Channel Flow (Launder, Reece and Rodi, 1975)

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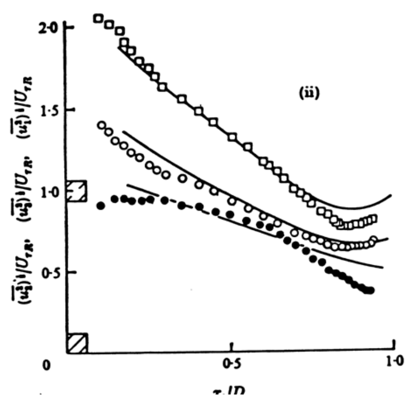

Turbulence Intensity

Algebraic Stress Model (Rodi, ZAMM 56, 1976)

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Stress Transport Model

$$\frac{d}{dt}\overline{u_i'u_j'} = \underbrace{c_s\frac{\partial}{\partial x_k}\left(\frac{k}{\varepsilon}\overline{u_k'u_m'}\frac{\partial}{\partial x_m}\overline{u_i'u_j'}\right)}_{\text{Diffusion}} \underbrace{-\overline{u_i'u_k'}\frac{\partial U_j}{\partial x_k} - \overline{u_j'u_k'}\frac{\partial U_i}{\partial x_k}}_{\text{Production}}$$
$$\underbrace{-c_1\frac{\varepsilon}{k}\left(\overline{u_i'u_j'} - \delta_{ij}\frac{2}{3}k\right) - \gamma\left(P_{ij} - \delta_{ij}\frac{2}{3}P\right)}_{\text{Pressure-Strain}} \underbrace{-\frac{2}{3}\delta_{ij}\varepsilon}_{\text{Dissipation}}$$

k-Equation

$$\frac{dk}{dt} = \underbrace{c_s\frac{\partial}{\partial x_k}\left(\frac{k}{\varepsilon}\overline{u_k'u_m'}\frac{\partial k}{\partial x_m}\right)}_{\text{Diffusion}} \underbrace{-\overline{u_k'u_m'}\frac{\partial U_k}{\partial x_m}}_{\text{Production}} \underbrace{-\varepsilon}_{\text{Dissipation}}$$

Algebraic Stress Model (Rodi, ZAMM 56, 1976)

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Rodi's Assumption

$$\frac{d}{dt}\overline{u_i'u_j'} - D_{ij} = \frac{\overline{u_i'u_j'}}{k}\left(\frac{dk}{dt} - D\right) = \frac{\overline{u_i'u_j'}}{k}(P-\varepsilon)$$

$$D = \frac{\partial}{\partial x_k}\left(\frac{k}{\varepsilon}\overline{u_k'u_l'}\frac{\partial k}{\partial x_l}\right) \qquad D_{ij} = \frac{\partial}{\partial x_k}\left(\frac{k}{\varepsilon}\overline{u_k'u_l'}\frac{\partial}{\partial x_l}\overline{u_i'u_j'}\right)$$

$$P = \overline{u_k'u_l'}\frac{\partial U_k}{\partial x_l} \qquad P_{ij} = \overline{u_i'u_k'}\frac{\partial U_j}{\partial x_k} - \overline{u_j'u_k'}\frac{\partial U_i}{\partial x_k}$$

Algebraic Stress Model (Rodi, ZAMM 56, 1976)

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$$\overline{u_i'u_j'} = k\left[\frac{2}{3}\delta_{ij} + \frac{1-\gamma}{c_1}\frac{\frac{P_{ij}}{\varepsilon} - \frac{2}{3}\frac{P}{\varepsilon}\delta_{ij}}{1 + \frac{1}{c_1}\left(\frac{P}{\varepsilon} - 1\right)}\right]$$

Simple Shear Flow

$$\nu_T = c_\mu\frac{k^2}{\varepsilon} \qquad c_\mu = \frac{2}{3}\frac{(1-\gamma)}{c_1}\frac{\left[1 - \frac{1}{c_1}\left(1 - \gamma\frac{P}{\varepsilon}\right)\right]}{\left[1 + \frac{1}{c_1}\left(\frac{P}{\varepsilon} - 1\right)\right]^2}$$

Conclusions

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- **Available models can predict the mean flow properties with reasonable accuracy.**
- **First-order modeling is reasonable when turbulence has a single length and velocity scale.**
- **The k-ε model gives reasonable results when a scalar eddy viscosity is sufficient.**
- **The stress transport models have the potential to be most accurate.**

Deficiencies of Existing Models

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- Adjustments of coefficients are needed.
- The derivation of the models are arbitrary.
- There is no systematic method for improving a model when it loses its accuracy.
- Models for complicated turbulent flows are not available.
- Realizability and other fundamental principles are sometimes violated.

Turbulence Models

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Thank you!

Questions?

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