ME 639 - Turbulence

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Turbulence Modeling

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Turbulence Modeling

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Outline

- Introduction to Turbulence Modeling
- Eddy Viscosity Models
- Prandtl Mixing Length Model
- One Equation Model
- Two-Equation Model
- k-ε Model
- Stress Transport Model

Two-Equation Turbulence Models

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One Length Scale ⊕ One Velocity Scale → First Order Model

When turbulence is not characterized by one scale, second order modeling has to be used.

Two-Equation Turbulence Models

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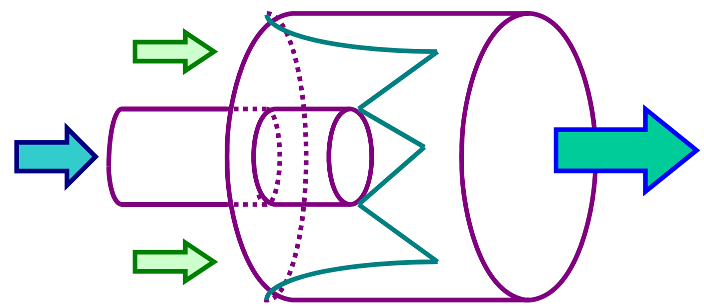
Example of Multi-Scale Flows



Two-Equation Turbulence Models

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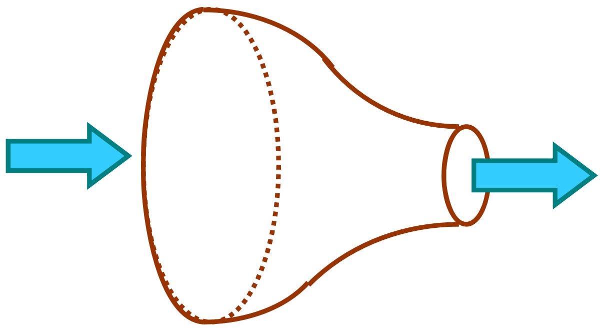
Example of Accelerated Flows



Prandtl-Kolmogorov Equation

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Eddy Viscosity → $\nu_T = k^{\frac{1}{2}}\ell$

Second Scale of turbulence → $z = k^m \ell^n$

Kolmogorov $f = \frac{k^{\frac{1}{2}}}{\ell} \qquad m = \frac{1}{2} \qquad n = -1$

Examples of z-Expression

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Launder-Spalding → $\varepsilon = \frac{k^{\frac{3}{2}}}{\ell} \qquad m = \frac{3}{2} \qquad n = -1$

Rotta → $k\ell \qquad m = 1 \qquad n = 1$

Rotta → $\ell \qquad m = 0 \qquad n = 1$

Spalding → $w = \frac{k}{\ell^2} \qquad m = 1 \qquad n = -2$

Thin Shear Layer Two-Equation Model

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$$\frac{dk}{dt} = \frac{\partial}{\partial y}\left(\frac{\nu_T}{\sigma_k}\frac{\partial k}{\partial y}\right) + k\left[\frac{\nu_T}{k}\left(\frac{\partial U}{\partial y}\right)^2 - c_D\frac{k}{\nu_T}\right]$$

$$\frac{dz}{dt} = \frac{\partial}{\partial y}\left(\frac{\nu_T}{\sigma_z}\frac{\partial z}{\partial y}\right) + z\left[C_1\frac{\nu_T}{k}\left(\frac{\partial U}{\partial y}\right)^2 - C_2\frac{k}{\nu_T}\right] + \frac{s_z}{\rho}$$

$\sigma_z, \; C_{1,} \; C_{1,} \; \sim$ Const.

Evaluation of Constants - Decaying Turbulence

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Decay Behind a Grid ⟷ $k \sim x^{-1}$

k-Eq.

$$U_0 \frac{dk}{dx} = -C_D \frac{k^2}{\nu_T}$$

z-Eq.

$$U_0 \frac{dz}{dx} = -C_2 \frac{kz}{\nu_T}$$

$$C_2 = C_D\left(m - \frac{n}{2}\right)$$

Evaluation of Constants - Inertial Sublayer

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$$\ell = \kappa y$$

$$k = C_D^{-\frac{1}{2}} u^{*2}$$

$$\nu_T = C_D^{-\frac{1}{2}} \kappa u^* y$$

$$z = k^m \ell^n = C_D^{-\frac{m}{2}} u^{*2m} (\kappa y)^n = A y^n$$

k-Eq. →

$$\frac{\nu_T}{k}\left(\frac{\partial U}{\partial y}\right)^2 - C_D \frac{k}{\nu_T} = 0$$

z-Eq. →

$$\frac{\partial}{\partial y}\left(\frac{\nu_T}{\sigma_z}\frac{\partial z}{\partial y}\right) + z\left[C_1 \frac{\nu_T}{k}\left(\frac{\partial U}{\partial y}\right)^2 - C_2 \frac{k}{\nu_T}\right] = 0$$

Evaluation of Constants - Inertial Sublayer

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Eliminating U

$$\frac{\partial}{\partial y}\left(\frac{\nu_T}{\sigma_z}\frac{\partial z}{\partial y}\right) + \frac{zk}{\nu_T}(C_1 C_D - C_2)$$

z-Eq. →

$$C_1 = \frac{C_2}{C_D} - \frac{\kappa^2 n^2}{\sigma_z C_D}$$

ε-Eq. →

$$C_2 = 2C_D$$

$$C_1 = 2 - \frac{\kappa^2}{\sigma_z C_D}$$

z-Equations

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$$\frac{D}{Dt}(k\ell) = \frac{\partial}{\partial y}\left(\frac{\nu_T}{\sigma_{k\ell}}\frac{\partial (k\ell)}{\partial y}\right) + 0.98 k^{\frac{1}{2}} \ell^2 \left(\frac{\partial U}{\partial y}\right)^2 - 0.059 k^{\frac{3}{2}} - \underbrace{\left[702\left(\frac{\ell}{y}\right)^6 k^{\frac{3}{2}}\right]}_{\text{For Near Wall Flows}}$$

$$\frac{D}{Dt} W = \frac{\partial}{\partial y}\left(\frac{\nu_T}{\sigma_W}\frac{\partial W}{\partial y}\right) + 1.04 W^{\frac{1}{2}} \left(\frac{\partial U}{\partial y}\right)^2 - 0.17 W^{\frac{3}{2}} + 3.5 \nu_T \left(\frac{\partial^2 U}{\partial y^2}\right)^2$$

$$\frac{D}{Dt}\varepsilon = \frac{\partial}{\partial y}\left(\frac{\nu_T}{\sigma_\varepsilon}\frac{\partial \varepsilon}{\partial y}\right) + 1.45 k \left(\frac{\partial U}{\partial y}\right)^2 - 0.18 \frac{\varepsilon^2}{k}$$

Boundary Conditions

Plan of Symmetry

$$\frac{\partial k}{\partial y}=0 \qquad \frac{\partial z}{\partial y}=0$$

Free Surface

$$U_0\frac{dk_0}{dx}=c_D\frac{k_0^2}{\nu_{To}} \qquad \frac{\partial k_0}{\partial y}=0$$

$$U_0\frac{dz_0}{dx}=c_z\frac{k_0 z_0}{\nu_{To}} \qquad \frac{\partial z_0}{\partial y}=0$$

Boundary Conditions

Near a Wall

$$U^+=\frac{1}{\kappa}\ln y^+ + C$$

$$k^+=C_D^{-\frac{1}{2}} \qquad z^+=C_D^{\frac{1}{2}(\frac{n}{2}-m)}(\kappa y^+)^n$$

Two-Equation Model

k-ε Model

Exact k-equation

$$\frac{dk}{dt}=-\frac{\partial}{\partial x_i}\left[\overline{u_i'\left(\frac{1}{2}u_j'u_j'+\frac{P'}{\rho}\right)}\right]-\overline{u_i'u_j'}\frac{\partial U_i}{\partial x_j}-\varepsilon+\nu\nabla^2 k$$

Exact ε-equation up to order $\left(\frac{u^3}{\Lambda\lambda^2}\right)$

$$\varepsilon=\nu\overline{\omega_i'\omega_i'}$$

$$\frac{d}{dt}\left(\frac{1}{2}\overline{\omega_i'\omega_i'}\right)=-\frac{1}{2}\frac{\partial}{\partial x_j}\left(\overline{u_j'\omega_i'\omega_i'}\right)+\overline{\omega_i'\omega_j'd_{ij}'}-\nu\overline{\frac{\partial\omega_i'}{\partial x_j}\frac{\partial\omega_i'}{\partial x_j}}$$

Two-Equation k-e Model

$$-\overline{u_i'u_j'}=\nu_T\left(\frac{\partial U_i}{\partial x_j}+\frac{\partial U_j}{\partial x_i}\right)-\frac{2}{3}k\delta_{ij} \qquad \nu_T=C_\mu\frac{k^2}{\varepsilon}$$

$$-\overline{u_i'\left(\frac{1}{2}u_j'u_j'+\frac{P'}{\rho}\right)}=\frac{\nu_T}{\sigma_k}\frac{\partial k}{\partial x_i}$$

Modeled k-Equation

$$\frac{dk}{dt}=\frac{\partial}{\partial x_i}\left(\frac{\nu_T}{\sigma_k}\frac{\partial k}{\partial x_i}\right)+\nu_T\left(\frac{\partial U_i}{\partial x_j}+\frac{\partial U_j}{\partial x_i}\right)\frac{\partial U_i}{\partial x_j}-\varepsilon$$

Two-Equation Model

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Modeling ε-Equation

Diffusion $\Rightarrow$ $$-\nu\overline{u_j'\omega_i'\omega_i'} = \frac{\nu_T}{\sigma_\varepsilon}\frac{\partial \varepsilon}{\partial x_j}$$

Production $$2\nu\overline{\omega_i'\omega_j' d_{ij}'} = c_{\varepsilon 1}\frac{\varepsilon}{k}\nu_T\left(\frac{\partial U_i}{\partial x_j}+\frac{\partial U_j}{\partial x_i}\right)\frac{\partial U_i}{\partial x_j}$$

Dissipation $\Rightarrow$ $$2\nu^2\overline{\frac{\partial \omega_i'}{\partial x_j}\frac{\partial \omega_i'}{\partial x_j}} = c_{\varepsilon 2}\frac{\varepsilon^2}{k}$$

Two-Equation Model

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ε-Equation

$$\frac{d\varepsilon}{dt} = \frac{\partial}{\partial x_j}\left(\frac{\nu_T}{\sigma_\varepsilon}\frac{\partial \varepsilon}{\partial x_j}\right) + c_{\varepsilon 1}\nu_T\frac{\varepsilon}{k}\left(\frac{\partial U_i}{\partial x_j}+\frac{\partial U_j}{\partial x_i}\right)\frac{\partial U_i}{\partial x_j} - c_{\varepsilon 2}\frac{\varepsilon^2}{k}$$

$c_\mu = 0.09$ $\quad c_{\varepsilon 1} = 1.45$ $\quad c_{\varepsilon 2} = 1.9$

$\sigma_k = 1$ $\quad \sigma_\varepsilon = 1.3$

Boundary Conditions Near a Wall

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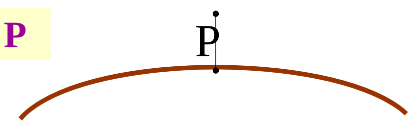
At a first Grid P



$$\frac{U_p}{u^{*2}}C_\mu^{1/4}k_P^{1/2} = \frac{1}{\kappa}\ln\left[Ey_p\frac{(C_\mu^{1/2}k_p)^{1/2}}{\nu}\right]$$

$$\int_0^{y_P}\varepsilon\, dy = C_\mu\frac{k_P^{3/2}}{\kappa}\ln\left[\frac{Ey_P(C_\mu^{1/2}k_P)^{1/2}}{\nu}\right]$$

Low Reynolds Number Models

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Jones-Launder

$$\frac{Dk}{Dt} = \frac{\partial}{\partial x_i}\left[\left(\nu+\frac{\nu_T}{\sigma_k}\right)\frac{\partial k}{\partial x_i}\right] + P - \hat{\varepsilon}$$

$$P = -\overline{u_i'u_j'}\frac{\partial U_i}{\partial x_j}$$

$$\frac{D\varepsilon}{Dt} = \frac{\partial}{\partial x_i}\left[\left(\nu+\frac{\nu_T}{\sigma_\varepsilon}\right)\frac{\partial \varepsilon}{\partial x_i}\right] + c_{\varepsilon 1}\frac{\hat{\varepsilon}}{k}P + 2\nu\nu_T\left(\frac{\partial^2 U_i}{\partial x_j \partial x_k}\right)^2 - c_{\varepsilon 2}\frac{\hat{\varepsilon}^2}{k}$$

$$\overline{u_i'u_j'} = \frac{2}{3}\delta_{ij}k - \nu_T\left(\frac{\partial U_i}{\partial x_j}+\frac{\partial U_j}{\partial x_i}\right)$$

Low Reynolds Number Models

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Jones-Launder

$$\nu_T = c_\mu \frac{k^2}{\varepsilon}$$

$$\hat{\varepsilon} = \varepsilon - 2\nu\left(\frac{\partial\sqrt{k}}{\partial x_j}\right)^2$$

$$c_\mu = 0.09\exp\left\{-\frac{2.51}{(1+R_T/50)}\right\}$$

$$R_T = \frac{k^2}{\nu\varepsilon}$$

$$c_{\varepsilon 2} = 1.9\left[1-0.3\exp\left(-R_T^2\right)\right]$$

Low Reynolds Number Models

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Saffman Model

$$\nu_T = \frac{Ak}{W}$$

$$\frac{D}{Dt}k = \alpha'' k\left(2S_{ij}^2\right)^{\frac{1}{2}} - kW + A''\frac{\partial}{\partial x_j}\left(\frac{k}{W}\frac{\partial k}{\partial x_j}\right)$$

$$\frac{D}{Dt}W^2 = \alpha' W^2\left[\eta U_{i,j}^2 + 2(1-\eta)S_{ij}^2\right]^{\frac{1}{2}} - \beta' W^3 + A'\frac{\partial}{\partial x_j}\left(\frac{k}{W}\frac{\partial W^2}{\partial x_j}\right)$$

$$\beta' = \frac{5}{3} \qquad A = \Gamma^2 \qquad \alpha'' = \Gamma \qquad \Gamma = \frac{u^{*2}}{k} \approx 0.3$$

$$\eta = 1 \qquad \alpha' = \frac{\beta'\alpha'' - 4A'\kappa^2}{A} \qquad A' = A'' = \frac{1}{2}A$$

Low Reynolds Number Models

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K-ω Model

$$\rho U\frac{\partial U}{\partial x} + \rho V\frac{\partial U}{\partial y} = \frac{\partial}{\partial y}\left[(\nu+\nu_T)\frac{\partial U}{\partial y}\right]$$

$$\nu_T = \frac{\alpha^* k}{\omega}$$

$$U\frac{\partial k}{\partial x} + V\frac{\partial k}{\partial y} = \nu_T\left(\frac{\partial U}{\partial y}\right)^2 - \beta^*\omega k + \frac{\partial}{\partial y}\left[(\nu+\sigma^*\nu_T)\frac{\partial k}{\partial y}\right]$$

$$U\frac{\partial \omega}{\partial x} + V\frac{\partial \omega}{\partial y} = \alpha\frac{\omega}{k}\nu_T\left(\frac{\partial U}{\partial y}\right)^2 - \beta\omega^2 + \frac{\partial}{\partial y}\left[(\nu+\sigma\mu_T)\frac{\partial \omega}{\partial y}\right]$$

Stress Transport - Boundary Layer

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Exact Transport Equation

$$\frac{D}{Dt}\overline{u'v'} = \underbrace{-\overline{v'^2}\frac{\partial U}{\partial y}}_{\text{Production}} \underbrace{-\frac{\partial}{\partial y}\left(\overline{u'v'^2} - \frac{\overline{P'u'}}{\rho}\right)}_{\text{Diffusion}}$$

$$\underbrace{+\overline{\frac{P'}{\rho}\left(\frac{\partial u'}{\partial y}+\frac{\partial u'}{\partial x}\right)}}_{\text{Pressure-Strain}} \underbrace{-2\nu\overline{\frac{\partial u'}{\partial x_k}\frac{\partial v'}{\partial x_k}}}_{\text{Dissipation}}$$

Stress Transport Model (Hanjalic)

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$$\text{Production} \approx k\frac{\partial U}{\partial y}$$

$$\text{Diffusion} \approx \frac{\partial}{\partial y}\left[\frac{\nu_T}{\sigma_T}\frac{\partial}{\partial y}\overline{u'v'}\right]$$

$$\text{Dissipation} \approx 0$$

$$\text{Pressure-strain} \approx \frac{k^{\frac{1}{2}}}{\ell}\overline{u'v'}$$

Modeled Equation

$$\frac{D}{Dt}\overline{u'v'} = \frac{\partial}{\partial y}\left[\frac{\nu_T}{\sigma_z}\frac{\partial}{\partial y}\overline{u'v'}\right] - c_\tau\left(k\frac{\partial U}{\partial y} + \frac{k^{\frac{1}{2}}}{\ell}\overline{u'v'}\right)$$

Second-Order Turbulence Models

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Concluding Remarks

- Two Equation Models
- k-ε Model
- Stress Transport Models
- Algebraic Stress Models

Turbulence Models

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Thank you!

Questions?