Prandtl's Mixing Length Hypothesis

The general form of the Boussineq eddy viscosity model is given as

$$-\overline{u_i' u_j'} = \nu_T \left(\frac{\partial U_i}{\partial x_j} + \frac{\partial U_j}{\partial x_i} \right) - \frac{2}{3} \delta_{ij} k, \qquad k = \frac{1}{2} \overline{u_m' u_m'}, \tag{1}$$

where ν_T is the eddy viscosity. For thin shear layer, the relevant component of (1) may be restated as

$$-\overline{u'v'} = \nu_T \frac{\partial U}{\partial y}. \tag{2}$$

Prandtl assumed that $\nu_T \sim u \ell_m$, where u is a turbulent velocity scale and ℓ_m is referred to as the mixing length. Furthermore, Prnadtl postulated that

$$u \sim \ell_m \left| \frac{\partial U}{\partial y} \right|, \tag{3}$$

and hence

$$\nu_T = \ell_m^2 \left| \frac{\partial U}{\partial y} \right|, \qquad -\overline{u'v'} = \ell_m^2 \left| \frac{\partial U}{\partial y} \right| \frac{\partial U}{\partial y}. \tag{4}$$

The mixing length ℓ_m depends on the nature of the flow and, in general, is space dependent.

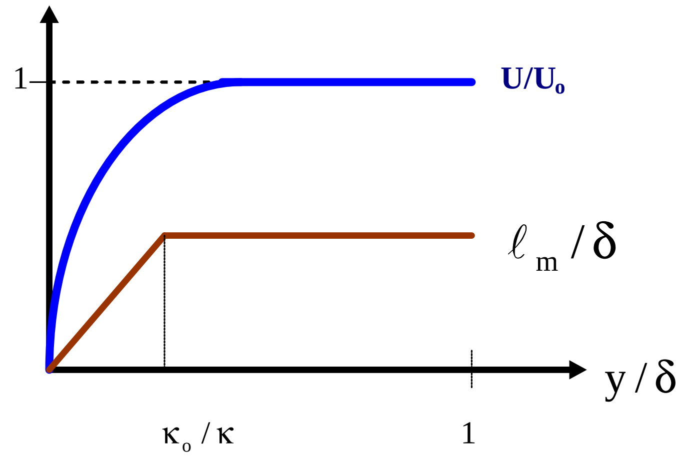
For free shear flows, ℓ_m is proportional to the half-width of the shear layer ℓ. For different flows, the mixing lengths are given as

$$\begin{aligned} \ell_m &= 0.09\ell, && \text{for plane jet,} \\ \ell_m &= 0.075\ell, && \text{for circular jet,} \\ \ell_m &= 0.07\ell, && \text{for mixing layer.} \end{aligned} \tag{5}$$

For boundary layer flows, several different forms were suggested. For instance, Escudier (1966, Imperial College Report) assumed

$$\ell_m = \begin{Bmatrix} \kappa y & \frac{y}{\delta} \le \frac{\kappa_0}{\kappa} \\ \kappa_0 \delta & \frac{y}{\delta} \ge \frac{\kappa_0}{\kappa} \end{Bmatrix}. \tag{6}$$

where $\kappa = 0.41$ and $\kappa_0 = 0.09$.

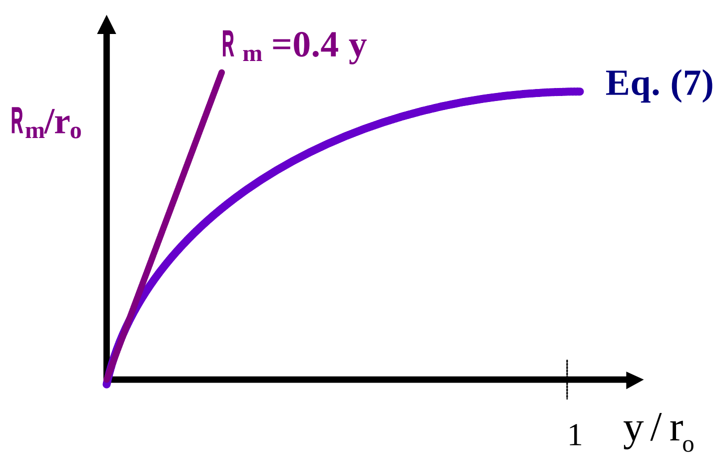


Variation of mean velocity and mixing length
in a turbulent boundary layer

For pipe flows, Nikuradse suggested the following expression:

$$\frac{\ell_m}{r_0} = 0.14 - 0.08\left(1 - \frac{y}{r_0}\right)^2 - 0.06\left(1 - \frac{y}{r_0}\right)^4, \tag{7}$$

where r_0 is the radius of the pipe.



Variation of mixing length in a turbulent pipe
flow according to Nikuradse.

Shortcomings of Mixing Length Hypothesis

The eddy viscosity of the mixing length model as is given by equation (4) implies that

$$\nu_T = 0 \qquad \text{for} \qquad \frac{\partial U}{\partial y} = 0. \tag{8}$$

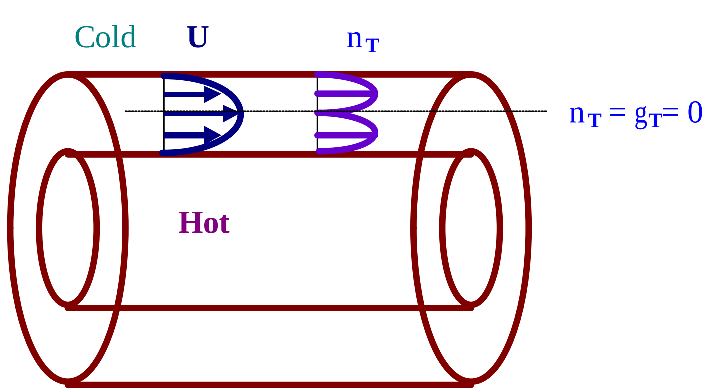
However, Equation (8) contradicts the experimental data in that ν_T, in general, is not zero when $\frac{\partial U}{\partial y} = 0$. For example, it is observed that $\nu_T \approx 0.8\nu_T|_{max}$ at the center of a pipe where $\frac{\partial U}{\partial y}$ vanishes.

Note that equation (8) usually does not introduce significant errors for momentum transport. This is because the turbulent stress approaches zero as $\frac{\partial U}{\partial y} \to 0$. However, the mixing length hypothesis also implies that

$$\gamma_T \approx \nu_T = \ell_m \left|\frac{\partial U}{\partial y}\right|, \tag{9}$$

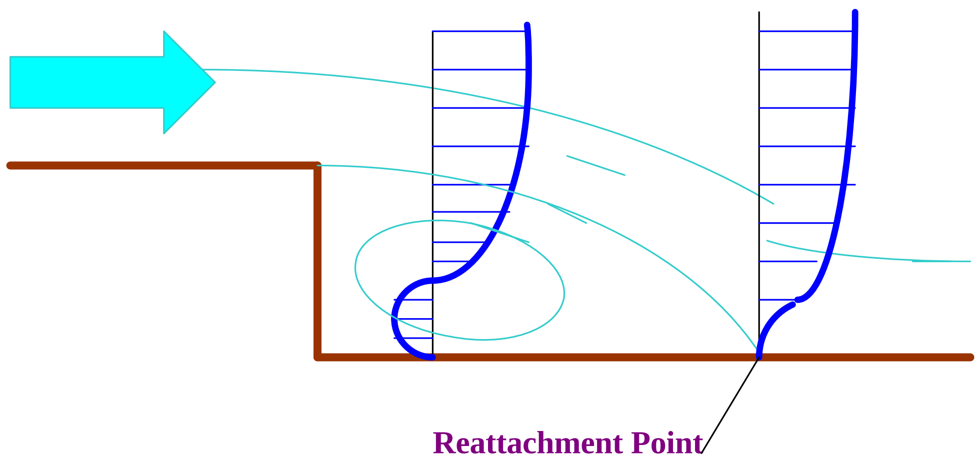
where γ_T is the heat or mass diffusivity. When $\frac{\partial U}{\partial y} = 0$, equation (9) implies that $\gamma_T = 0$. This result, however, produces major errors in engineering heat transfer calculations.

The figure shows an example of a heat exchanger where large resistance to heat flow is predicted by the mixing length model but never observed experimentally.



Variations of mean velocity and eddy viscosity in a heat exchanger.

Another example of failure of the mixing length theory is illustrated in the recirculating flow shown in the figure:



Schematics of backward facing step flows.

Experimental data indicates that the maximum heat flux that occurs at the reattachment point, while the mixing length model leads to $\gamma_T = \nu_T = 0$ (since $\frac{\partial U}{\partial y} = 0$).

One-Equation Turbulence Models

Many of the one and multi-equation turbulence models are based on the Prandtl-Kolmogorov equation given by

$$\nu_T = k^{\frac{1}{2}} \ell , \tag{10}$$

where $k = \frac{1}{2}\overline{u_i' u_i'}$ is the kinetic energy of turbulence and ℓ is a turbulent length scale. It is expected that $k^{\frac{1}{2}}$ be a better representative of the turbulent velocity scale when compared with $\ell \left| \frac{\partial U}{\partial y} \right|$.

The general equation for dynamics of k is given as

$$\frac{d}{dt} k = -\frac{\partial}{\partial x_j}\left[\overline{u_j'\left(\frac{u_i' u_i'}{2} + \frac{P'}{\rho}\right)}\right] - \overline{u_i' u_j'}\frac{\partial U_i}{\partial x_j} - \nu \overline{\frac{\partial u_i'}{\partial x_j}\frac{\partial u_i'}{\partial x_j}} + \nu \frac{\partial^2 k}{\partial x_j \partial x_j} . \tag{11}$$

For a thin shear layer, equation (11) may be restated as

$$\frac{dk}{dt} = -\frac{\partial}{\partial y}\left[\overline{v'(\frac{u_i'u_i'}{2} + \frac{P'}{\rho})}\right] - \overline{u'v'}\frac{\partial U}{\partial y} - \varepsilon\,, \tag{12}$$

where the viscous diffusion is neglected and ε is the dissipation. Equation (12) clearly shows that the kinetic energy of turbulence is convected, diffused, produced, and dissipated.

Modeling k-Equation

To close the k-equation, the unknown terms in equation (12) must be modeled. We assume

$$-\overline{u'v'} = \nu_T \frac{\partial U}{\partial y}\,, \tag{13}$$

where ν_T is given by (10). The diffusion term in (12) is modeled by a gradient law. i.e.,

$$-\overline{v'\left(\frac{1}{2}u_i'u_i' + \frac{P'}{\rho}\right)} = \frac{\nu_T}{\sigma_k}\frac{\partial k}{\partial y}\,, \tag{14}$$

where σ_k is a constant Prandtl number for k.

The dissipation is given as

$$\varepsilon = c_D \frac{k^{\frac{3}{2}}}{\ell} \tag{15}$$

Therefore, the closed k-equation becomes

$$\frac{dk}{dt} = \frac{\partial}{\partial y}\left(\frac{\nu_T}{\sigma_k}\frac{\partial k}{\partial y}\right) + \nu_T\left(\frac{\partial U}{\partial y}\right)^2 - c_D \frac{k^{\frac{3}{2}}}{\ell}\,. \tag{16}$$

Here, σ_k and c_D are constants and ℓ is a length scale distribution.

Near Wall Distribution

Near a wall, the convection and diffusion of turbulent kinetic energy may be neglected. The production must then be balanced by the dissipation, and Equation (16) then becomes

$$\nu_T\left(\frac{\partial U}{\partial y}\right)^2 = c_D\frac{k^{\frac{3}{2}}}{\ell} \qquad \text{or} \qquad \frac{(\overline{u'v'})^2}{\nu_T} = c_D\frac{k^{\frac{3}{2}}}{\ell} \tag{17}$$

Using (10), we find

$$c_D = \frac{(\overline{u'v'})^2}{k^2}\,. \tag{18}$$

From the experimental results, one finds $-\frac{\overline{u'v'}}{k} \approx 0.25 \sim 0.3$.

Thus,

$$c_D \approx 0.07 \sim 0.08\,. \tag{19}$$

For turbulence kinetic energy Prandtl number,

$$\sigma_k = 1 \tag{20}$$

is usually used.

Distribution of Length Scale

The distribution of the length scale ℓ is needed to complete the system of governing equations. Near a wall, from (18), we find

$$\left|\overline{u'v'}\right| = c_D^{\frac{1}{2}}k\,. \tag{21}$$

Using (10) and (13), we find

$$k^{\frac{1}{2}} = c_D^{-\frac{1}{2}}\ell\left|\frac{\partial U}{\partial y}\right|, \qquad -\overline{u'v'} = c_D^{-\frac{1}{2}}\ell^2\left(\frac{\partial U}{\partial y}\right)^2 \tag{22}$$

Also, we know that in the inertial sublayer,

$$\frac{\partial U}{\partial y} = \frac{u^*}{\kappa y}, \qquad -\overline{u'v'} = u^{*2}. \tag{23}$$

Using (23) in (22) and solving for ℓ, we find

$$\ell = c_D^{\frac{1}{4}} \kappa y = c_D^{\frac{1}{4}} \ell_m \tag{24}$$

For $c_D = 0.08$, $\kappa = 0.4$, we find

$$\ell = 0.2y \tag{25}$$

It was suggested that to use equation (24) in the entire flow domain with ℓ_m given by an empirical mixing length expression. Nikuradse's equation given by (7) has been used extensively. In immediate vicinity of the wall, further modification for viscous effects are needed.

Achievements of the Model

Heat transfer in the heat exchanger and separated flows are predicted with reasonable accuracy.

Shortcomings

- Transport of the turbulent length scale is not accounted for.
- The model offers little advantage over the mixing length model.