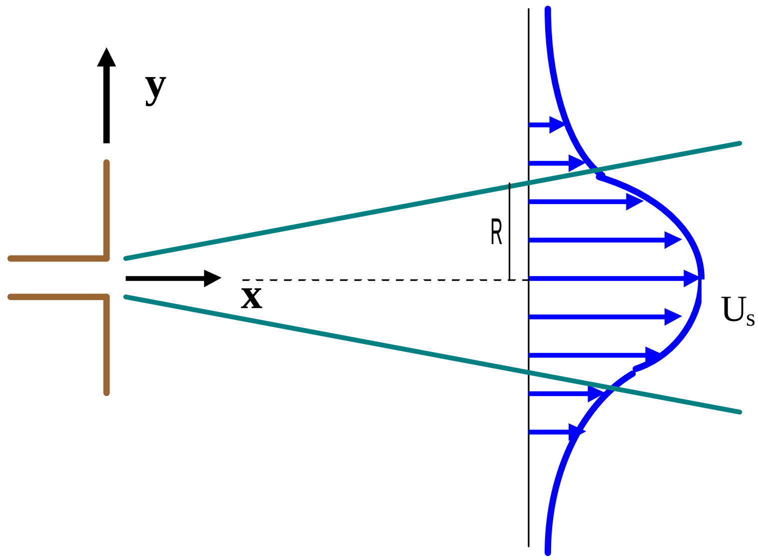
Turbulent Plane Jet Flow

Turbulent jet flow shown schematically in the figure is studied in this section.



Equation of Motion

$$U\frac{\partial U}{\partial x} + V\frac{\partial U}{\partial y} + \frac{\partial}{\partial y}\overline{u'v'} = 0 \quad (1)$$

Momentum Integral

$$\int_{-\infty}^{+\infty} dy\left[\frac{\partial}{\partial x}\left(U^2\right) + \frac{\partial}{\partial y}\left(UV\right) + \frac{\partial}{\partial y}\overline{u'v'}\right] = 0 \quad (2)$$

or

$$\frac{d}{dx}\int_{-\infty}^{+\infty} U^2 dy = 0\,. \quad (3)$$

Integrating Equation (3) we find

$$\rho\int_{-\infty}^{+\infty} U^2 dy = M = \text{Jet Momentum} \quad (4)$$

That is the total momentum of the jet is conserved.

Continuity Equation

$$\frac{\partial U}{\partial x}+\frac{\partial V}{\partial y}=0 \tag{5}$$

Introducing the stream function, ψ,

$$U=\frac{\partial \psi}{\partial y}, \qquad V=-\frac{\partial \psi}{\partial x} \tag{6}$$

the continuity equation given by (5) is identically satisfied.

Self similar solutions for the mean velocity and turbulent stress are given as

$$U=U_s f(\xi), \quad -\overline{u'v'}=U_s^2 g(\xi), \qquad \xi=\frac{y}{\ell} \tag{7}$$

We assume

$$U_s=cx^m, \qquad \ell=Dx^n. \tag{8}$$

Equation (1) then implies, $\quad 2m-1=2m-n, \quad$ i.e. $n=1$

Equation (4) implies. $\quad 2m+n=0, \quad$ i.e. $\quad m=-\frac{1}{2}$

Hence,

$$U_s=Cx^{-\frac{1}{2}}, \qquad \ell=Dx. \tag{9}$$

Now,

$$x=\frac{y}{\ell}=\frac{y}{Dx}, \quad \frac{\partial x}{\partial x}=-\frac{y}{Dx^2}=-\frac{x}{x}, \quad \frac{\partial x}{\partial y}=\frac{1}{\ell}=\frac{1}{Dx} \tag{10}$$

Integrating (6), we find

$$y=\ell U_s\int_0^x f(x_1)dx_1 \tag{11}$$

Let

$$F(\xi)=\int_0^{\xi} f(\xi_1)d\xi_1 \tag{12}$$

Then,

$$\psi = CDx^{\frac{1}{2}}F(\xi) \tag{13}$$

$$-\overline{u'v'} = C^2x^{-1}g(\xi) \tag{14}$$

Now

$$V=-\frac{\partial\psi}{\partial x}=-CD\left[\frac{1}{2}x^{-\frac{1}{2}}F(\xi)+x^{\frac{1}{2}}F'(\xi)\left(-\frac{\xi}{x}\right)\right]=-CDx^{-\frac{1}{2}}\left(\frac{1}{2}F-\xi F'\right) \tag{15}$$

$$U = Cx^{-\frac{1}{2}}F'(\xi) \tag{14}$$

$$\frac{\partial U}{\partial x}=-\frac{1}{2}Cx^{-\frac{3}{2}}F'+cx^{-\frac{1}{2}}F''\left(-\frac{\xi}{x}\right)=-Cx^{-\frac{3}{2}}\left(\frac{1}{2}F'+\xi F''\right) \tag{15}$$

$$\frac{\partial U}{\partial y}=Cx^{-\frac{1}{2}}F''(\xi)\frac{1}{Dx}=\frac{C}{D}x^{-\frac{3}{2}}F'' \tag{16}$$

$$-\frac{\partial\overline{u'v'}}{\partial y}=\frac{C^2}{D}x^{-2}g' \tag{17}$$

Using (13) – (17), Equation (1) becomes

$$-Cx^{-\frac{1}{2}}F'\left(-Cx^{-\frac{3}{2}}\right)\left(\frac{1}{2}F'+\xi F''\right)-CDx^{-\frac{1}{2}}\left(\frac{1}{2}F-\xi F'\right)\frac{C}{D}x^{-\frac{3}{2}}F''=\frac{C^2}{D}x^{-2}g'. \tag{18}$$

Simplifying (18), it follows that

$$-F'\left(\frac{1}{2}F'+xF''\right)-\left(\frac{1}{2}F-xF'\right)F''=\frac{1}{D}g' \tag{19}$$

or

$$F'^2 + FF'' = -\frac{2}{D} g' \tag{20}$$

Introducing the eddy viscosity,

$$-\overline{u'v'} = \nu_T \frac{\partial U}{\partial y} \tag{21}$$

or

$$\nu_T = -\frac{\overline{u'v'}}{\frac{\partial U}{\partial y}} = \frac{U_s^2 g(\xi)}{\frac{U_s}{\ell} F''} = U_s \ell \frac{g}{F''} \tag{22}$$

That is

$$\frac{\nu_T}{U_s \ell} = \frac{1}{R_T} = \frac{g}{F''} \tag{23}$$

or

$$g = \frac{1}{R_T} F'', \qquad R_T \approx 25.7 \tag{24}$$

Equation (20) now becomes

$$F'^2 + FF'' = -F''' \tag{25}$$

where we assumed $\frac{2}{DR_T} = 1$ and $D = 0.078$. Equation (25) may be restated as

$$(FF')' + F''' = 0. \tag{26}$$

Integrating Equation (26), we find

$$FF' + F'' = c_1 \tag{27}$$

Boundary conditions for the jet flow are

$$F'(0) = 1, \; F'(\infty) = 0, \; F(0) = 0, \; (\text{or } F''(\infty) = 0) \tag{28}$$

Hence, $c_1 = 0$, and integrating Equation (27), we find

$$\frac{1}{2}F^2 + F' = c_2 = 1 \text{ or } \frac{2dF}{2-F^2} = d\xi,\; 2\frac{1}{\sqrt{2}}\tan^{-1}\frac{F}{\sqrt{2}} = x + c_3,\; c_3 = 0. \quad (29)$$

Therefore,

$$F = \sqrt{2}\tanh\frac{\xi}{\sqrt{2}}, \quad (30)$$

and

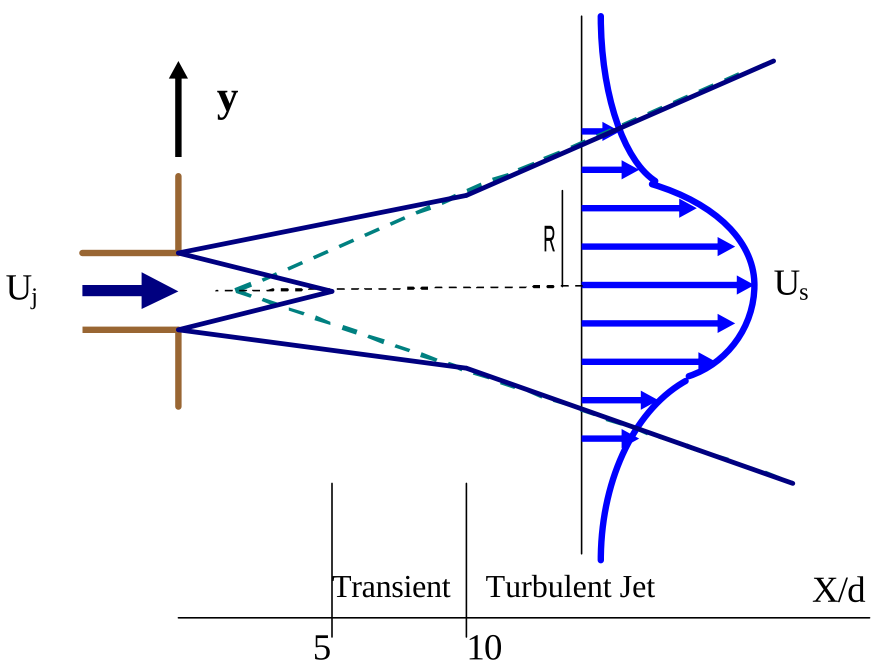
$$f = F' = \frac{1}{\cos^2\frac{\xi}{\sqrt{2}}} = \operatorname{sech}^2\frac{\xi}{\sqrt{2}} \quad (31)$$

If d is the width of the jet, then

$$\frac{U_s}{U_j} = 2.7\left(\frac{d}{x}\right)^{\frac{1}{2}}, \qquad \ell = 0.078x \quad (32)$$

where

$$\int_{-\infty}^{+\infty} U^2 dy = U_j^2 d \quad (33)$$

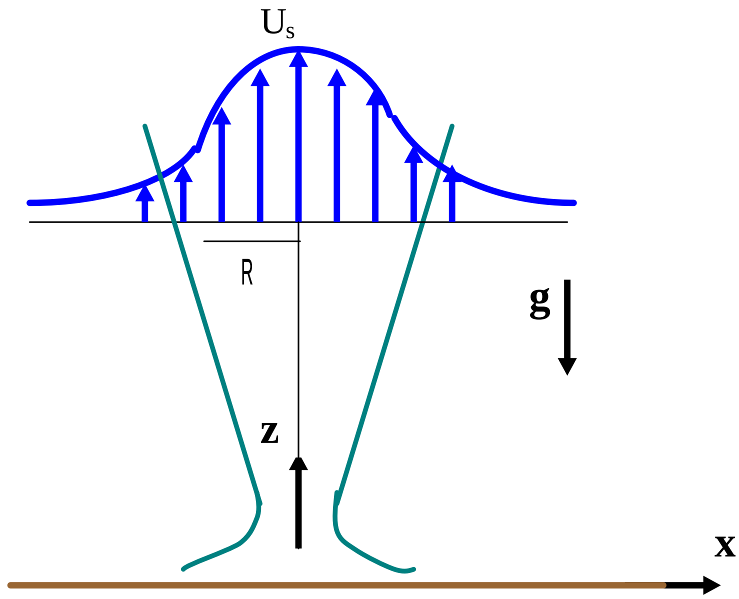


Thermal Plumes

The equations governing the motion and heat transfer turbulent flows are given as

$$\begin{cases} U_j \dfrac{\partial U_i}{\partial x_j} + \dfrac{\partial}{\partial x_j}\overline{u_i' u_j'} = -\dfrac{1}{\rho}\dfrac{\partial P}{\partial x_i} + \nu \dfrac{\partial^2 U_i}{\partial x_j \partial x_j} + \dfrac{g_0 \overline{T}}{T_0}\delta_{ij}, \qquad \dfrac{\partial U_i}{\partial x_i} = 0 \\ U_j \dfrac{\partial \overline{T}}{\partial x_j} + \dfrac{\partial}{\partial x_j}\overline{T' u_j'} = \gamma \dfrac{\partial^2 \overline{T}}{\partial x_j \partial x_j} \end{cases} \tag{1}$$

The special case of two-dimensional thermal plumes is described in the following section.



Two Dimensional Plane Plumes

For a two a two-dimensional plume as shown in the figure, the equation of balance of moment, heat flow and continuity are given as:

$$U \frac{\partial W}{\partial x} + W \frac{\partial W}{\partial z} + \frac{\partial}{\partial x}\overline{u' w'} = \frac{g_0 \overline{T}}{T_0} \tag{2}$$

$$U \frac{\partial \overline{T}}{\partial x} + W \frac{\partial \overline{T}}{\partial z} + \frac{\partial}{\partial x}\overline{T' u'} = 0 \tag{3}$$

$$\frac{\partial U}{\partial x} + \frac{\partial W}{\partial z} = 0 \tag{4}$$

In addition the heat flux integral given by

$$\int_{-\infty}^{+\infty} \overline{T} W dx = \text{const} = \frac{q}{\rho c_p} \tag{5}$$

Introducing the similarity variables

$$x = \frac{x}{\ell} \tag{6}$$

the following self similar solutions may be assumed:

$$W = U_s f(\xi), \quad -\overline{u'w'} = U_s^2 g(\xi), \qquad \overline{T} = T_s \theta(\xi), \qquad -\overline{T'u'} = T_s U_s h(\xi) \tag{7}$$

Now let

$$U_s = Az^m, \qquad \ell = Bz^n, \qquad T_s = cz^p \tag{8}$$

Equation (2) implies that

$$2m - 1 = 2m - n = p \quad \text{or} \qquad n = 1 \quad \text{and} \quad p = 2m - 1 \tag{9}$$

Equation (5) implies that

$$m + p + n = 0 \qquad \text{or} \qquad m + p = -1 \tag{10}$$

Equations (9) and (10) imply that

$$m = 0 \quad \text{and} \quad p = -1. \tag{11}$$

Hence,

$$U_s = \text{const}, \qquad \ell = Bz, \qquad T_s = Cz^{-1}. \tag{12}$$

Now

$$\xi = \frac{x}{\ell} = \frac{x}{Bz}, \qquad \frac{\partial \xi}{\partial x} = \frac{1}{Bz}, \qquad \frac{\partial \xi}{\partial z} = -\frac{\xi}{z} \tag{13}$$

Introducing the stream function

$$\psi = \ell U_s F(\xi), \qquad F' = f, \qquad \psi = U_s BzF(\xi) \tag{14}$$

$$W = U_s F', \qquad U = -\frac{\partial\psi}{\partial z} = -U_s B(F - \xi F'), \qquad \frac{\partial W}{\partial x} = \frac{U_s}{Bz} F'' \tag{15}$$

$$\frac{\partial W}{\partial z} = -\frac{U_s \xi}{z} F'', \qquad T = Cz^{-1}\theta(\xi), \qquad \frac{\partial T}{\partial x} = Cz^{-2}\theta', \qquad \frac{\partial T}{\partial z} = -Cz^{-2}(\theta + \xi\theta'), \tag{16}$$

$$-\frac{\partial \overline{u'w'}}{\partial x} = \frac{U_s^2}{Bz} g', \qquad -\frac{\partial \overline{T'u'}}{\partial x} = \frac{U_s C}{B} z^{-2} h'. \tag{17}$$

Using (14)-(17) in (2) and (3), we find

$$-U_s B(F - \xi F')\frac{U_s}{Bz}F'' + U_s F'\left(-\frac{U_s \xi}{z}\right)F'' = \frac{U_s^2}{Bz} g' + \frac{g_0 C}{T_0 z}\theta \tag{18}$$

$$-U_s B(F - \xi F')\frac{C}{B} z^{-2}\theta' + U_s F'(-cz^{-2})(\theta + \xi\theta') = \frac{U_s C}{B} z^{-2} h' \tag{19}$$

Simplifying Equations (18) and (19), we find

$$-FF'' = \frac{1}{B} g' + \frac{g_0 C}{T_0 U_s^2}\theta \tag{20}$$

$$-(F\theta)' = \frac{1}{B} h' \tag{21}$$

Introducing the eddy viscosity,

$$\nu_T = \frac{-\overline{u'w'}}{\frac{\partial W}{\partial x}} = -\frac{\overline{\theta' u'}}{\frac{\partial T}{\partial x}} = U_s \ell \frac{g}{F''} = U_s \ell \frac{h}{\theta'} \tag{22}$$

or

$$g = \frac{1}{R_T} F'', \qquad h = \frac{1}{R_T}\theta' \tag{23}$$

Now letting

$$B = \frac{1}{R_T} \text{ and } \frac{g_0 C}{T_0 U_s^2} = 1 \tag{24}$$

Then

$$FF'' + F''' + q = 0 \tag{25}$$

$$F\theta + \theta' = \text{const} = 0 \tag{26}$$

The corresponding boundary conditions are:

$$F'(0) = 1, \qquad F(0) = 0, \qquad F'(\infty) = 0, \qquad \theta(0) = 1, \tag{27}$$

An exact solution is not available. Here an approximate solution is presented. In the neighborhood of $x = 0$,

$$F \approx \xi + \dots \tag{28}$$

It then follows that

$$\frac{\theta'}{\theta} = -\xi, \qquad \ln\theta \approx -\frac{\xi^2}{2} + \ln c, \qquad \theta \approx e^{-\frac{\xi^2}{2}} \tag{29}$$

Equation (25) may then be approximated as

$$F''' + \xi F'' = -e^{-\frac{\xi^2}{2}}, \tag{30}$$

The solution to (30) is

$$F'' = -\xi e^{-\frac{\xi^2}{2}} \tag{31}$$

Integrating (30), it follows that

$$f = F' = e^{-\frac{\xi^2}{2}} \tag{32}$$

Therefore, the mean velocity and temperature fields in the plume are given by

$$\frac{W}{U_s} = e^{-\frac{\xi^2}{2}}, \qquad \frac{T}{T_s} = e^{-\frac{\xi^2}{2}}. \tag{33}$$