From Chaos to Turbulence

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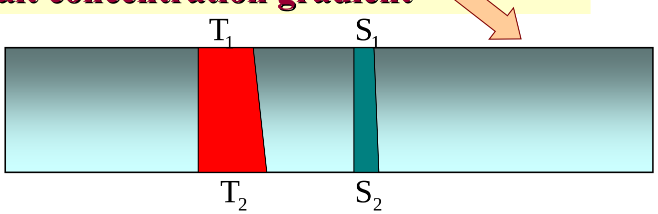
Outline

- Double Diffusive Convection
- Thermal Convection
- Isotropic Turbulence
- Bifurcation
- Turbulence

Chaos in a Double-Diffusive Convection Model

S. A. Abu-Zaid and G. Ahmadi,
Appl. Math. Modeling, Vol. 13 (1989)

Fluid Layer heat from below with a salt concentration gradient



Chaos in a Double-Diffusive Convection Model

Governing Equations

$$\rho_0(\partial_t \mathbf{u} + \mathbf{u}\cdot\nabla\mathbf{u}) = -\nabla(p - p_0) + g(\rho - \rho_0) + \rho\nu\nabla^2\mathbf{u}$$

$$\partial_t T + \mathbf{u}\cdot\nabla T = \kappa_T \nabla^2 T$$

$$\nabla\cdot\mathbf{u} = 0$$

$$\partial_t S + \mathbf{u}\cdot\nabla S = \kappa_S \nabla^2 S$$

$$\rho = \rho_0(1 - \alpha(T - T_0) + \beta(S - S_0))$$

Chaos in a Double-Diffusive Convection Model

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Stream Function

$$\mathbf{u} = \left(-\partial_z\psi, 0, \partial_x\psi\right)$$

$$T - T_0 = \Delta T\left(1 - z + \theta(x,z,t)\right)$$
$$S - S_0 = \Delta S\left(1 - z + \Sigma(x,z,t)\right)$$

Nondimensional Governing Equations

$$\sigma^{-1}\left[\partial_t\nabla^2\psi + J\left(\psi,\nabla^2\psi\right)\right] = R_T\partial_x\theta - R_S\partial_x\Sigma + \nabla^4\psi$$

$$\partial_t\theta + J(\psi,\theta) = \partial_x\psi + \nabla^2\theta$$

$$\partial_t\Sigma + J(\psi,\Sigma) = \partial_x\psi + \tau\nabla^2\Sigma$$

Chaos

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Jacobian

$$J(\psi,\theta) = \partial_x\psi\partial_y\theta - \partial_y\psi\partial_x\theta$$

Thermal Rayleigh Number

$$R_T = \frac{g\alpha\Delta T d^3}{\kappa_T\nu}$$

Solutal Rayleigh Number

$$R_S = \frac{g\beta\Delta T d^3}{\kappa_S\nu}$$

Prandtl Number

$$\sigma = \frac{\nu}{\kappa_T}$$

Lewis Number

$$k = \frac{\kappa_s}{\kappa_T}$$

Modal Motions

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$$\psi = 2(2p)^{\frac{1}{2}}\frac{\lambda}{\pi}\sin\frac{\pi x}{\lambda}\sin\pi z\, X(t^*)$$

$$\theta = 2\left(\frac{2}{p}\right)^{\frac{1}{2}}\cos\frac{\pi x}{\lambda}\sin 2\pi z\, Y(t^*) - \frac{1}{\pi}\sin 2\pi z\, Z(t^*)$$

$$\Sigma = 2\left(\frac{2}{p}\right)^{\frac{1}{2}}\cos\frac{\pi x}{\lambda}\sin\pi z\, U(t^*) - \frac{1}{\pi}\sin 2\pi z\, V(t^*)$$

Modal Motions

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$$\dot{X} = \sigma\left(-X + r_T Y - r_S U\right)$$

$$r_T = \frac{\pi^2}{\lambda^2 p^3}R_T$$

$$\dot{Y} = -Y + X(1 - Z)$$

$$r_S = \frac{\pi^2}{\lambda^2 p^3}R_S$$

$$\dot{Z} = a\left(-Z + XY\right)$$

$$a = \frac{4\pi^2}{p}$$

$$\dot{U} = -kU + X(1 - V)$$

$$\dot{V} = a\left(-kV + XU\right)$$

Modal Amplitudes $r_T = 10$

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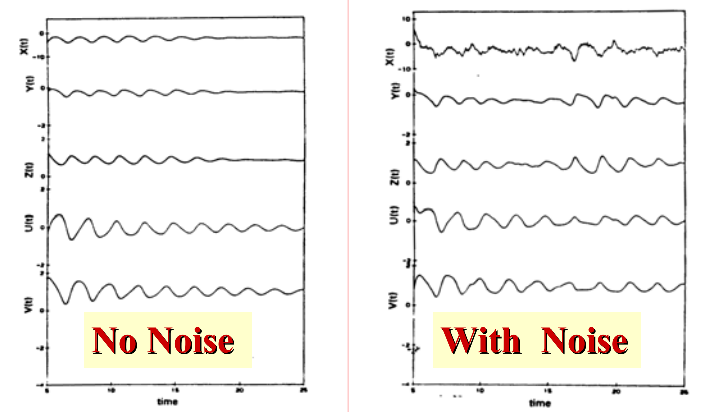

Modal Amplitudes $r_T = 40$

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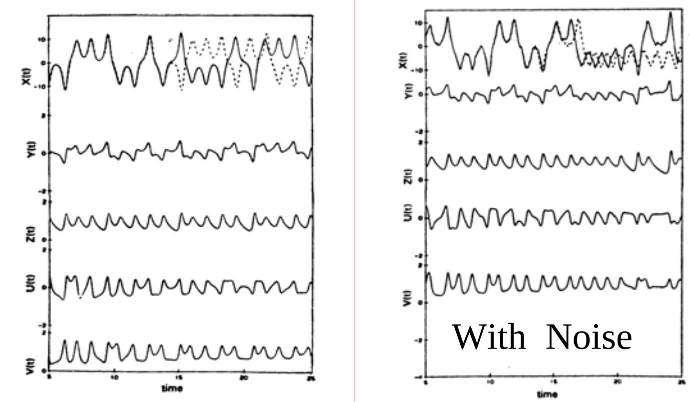

Phase Plane $r_T = 40$

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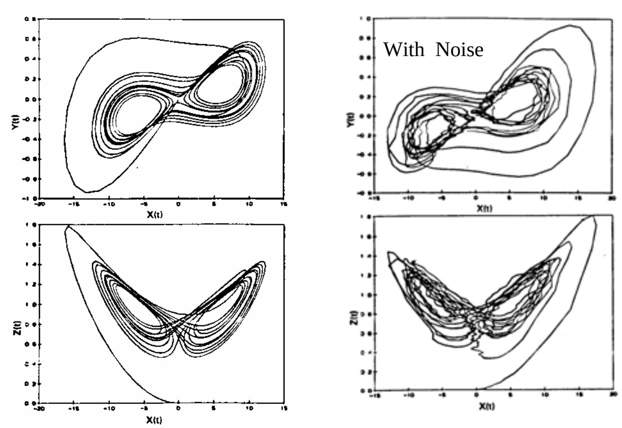

Autocorrelation Functions $r_T = 40$

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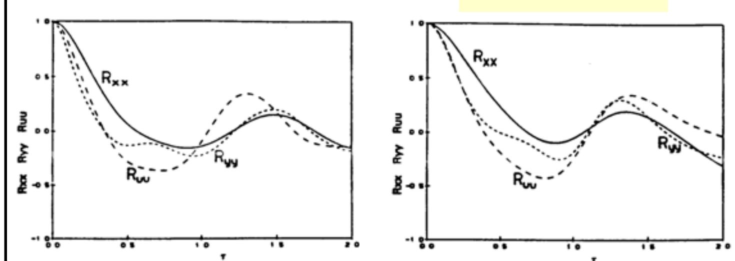
With Noise



Stream Function Modal Amplitudes

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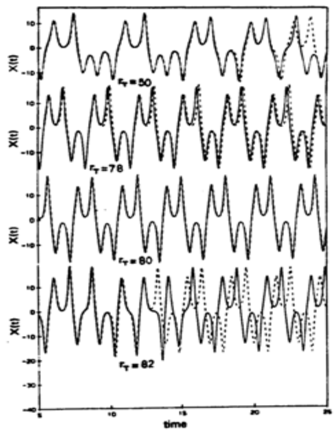
Different r_T



Stream Function Modal Amplitudes $r_T = 80$

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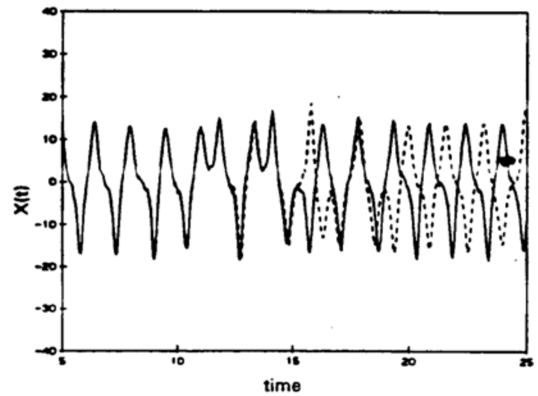

Poincare Map $r_T = 40$

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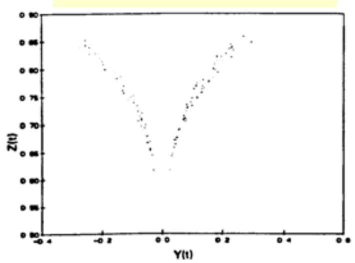
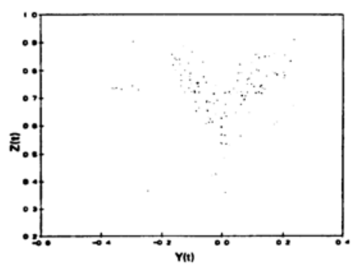
Without Noise



With Noise



Chaotic Thermal Convection

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(McLaughlin and Orszag, JFM, 1982)

Governing Equations

$$\frac{\partial \mathbf{v}}{\partial t} = \mathbf{v} \times \boldsymbol{\omega} - \nabla \pi + \Pr\left(\nabla^2 \mathbf{v} + \mathbf{k}\theta\right)$$

$$\frac{\partial \theta}{\partial t} + \mathbf{v} \cdot \nabla \theta = \mathrm{Ra}\, w + \nabla^2 \theta$$

$$\nabla \cdot \mathbf{v} = 0$$

Chaotic Thermal Convection

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Vorticity ⟷ $\boldsymbol{\omega} = \nabla \times \mathbf{v}$

Pressure Head ⟷ $\pi = p + \frac{1}{2}|v|^2$

Rayleigh Number ⟷ $Ra = \frac{g\beta H^3 \Delta T}{\nu\kappa}$

Prandtl Number ⟷ $Pr = \frac{\nu}{\kappa}$

Chaotic Thermal Convection

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Fourier-Chebyshev Series

$$\mathbf{v}(x,y,z,t) = \sum_{|m|<\frac{1}{2}M} \sum_{|n|<\frac{1}{2}N} \sum_{p=0}^{P} \tilde{\mathbf{v}}(m,n,p,t) e^{2\pi i\left(\frac{mx}{X}+\frac{ny}{Y}\right)} T_p(2z)$$

$$\theta(x,y,z,t) = \sum_{|m|<\frac{1}{2}M} \sum_{|n|<\frac{1}{2}N} \sum_{p=0}^{P} \tilde{\theta}(m,n,p,t) e^{2\pi i\left(\frac{mx}{X}+\frac{ny}{Y}\right)} T_p(2z)$$

Chaotic Thermal Convection

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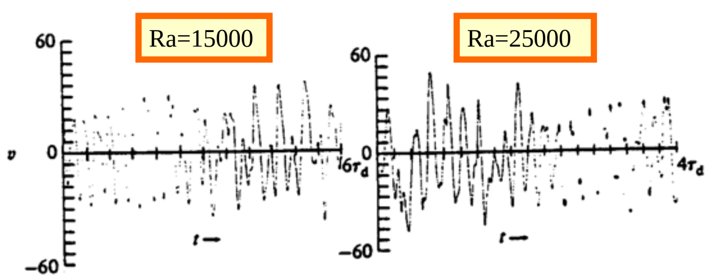

Two-D Turbulence

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(G. Ahmadi and V.W. Goldschmidt, Developments in Mechanics Vol. 6, 1971)

Navier-Stokes

$$\frac{\partial \mathbf{u}_f}{\partial t} + \mathbf{u}_f \cdot \nabla \mathbf{u}_f = -\frac{1}{\rho_f}\nabla P + \frac{1}{Re_L}\nabla^2 \mathbf{u}_f$$

$$\nabla \cdot \mathbf{u}_f = 0$$

Wave Number Space

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Fourier Series

$$\mathbf{u}_f(\mathbf{x},t)=\sum_{\mathbf{K}}\mathbf{u}(K,t)e^{i\mathbf{K}\cdot\mathbf{x}}$$

x-Component

$$\frac{\partial u}{\partial t}(\kappa_x,\kappa_y,t)=-\frac{K_x^2+K_y^2}{Re_L}u(K_x,K_y,t)$$
$$-i\left\{\begin{array}{l}K_x\left[1-\frac{K_x^2}{K_x^2+K_y^2}\right]\sum_{K_x^1}\sum_{K_y^1}u(K_x^1,K_y^1,t)u(K_x-K_x^1,K_y-K_y^1,t)\\+K_y\left[1-\frac{2K_x^2}{K_x^2+K_y^2}\right]\sum_{K_x^1}\sum_{K_y^1}u(K_x^1,K_y^1,t)v(K_x-K_x^1,K_y-K_y^1,t)\\+K_y\left[\frac{-K_xK_y}{K_x^2+K_y^2}\right]\sum_{K_x^1}\sum_{K_y^1}v(K_x^1,K_y^1,t)v(K_x-K_x^1,K_y-K_y^1,t)\end{array}\right\}$$

Wave Number Space

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y-Component

$$\frac{\partial}{\partial t}v(K_x,K_y,t)=-\left[\frac{K_x^2+K_y^2}{Re_L}\right]v(K_x,K_y,t)$$
$$-i\left\{\begin{array}{l}K_x\left[-\frac{K_xK_y}{K_x^2+K_y^2}\right]\sum_{K_x^1}\sum_{K_y^1}u(K_x^1,K_y^1,t)u(K_x-K_x^1,K_y-K_y^1,t)\\+K_x\left[1-\frac{2K_y^2}{K_x^2+K_y^2}\right]\sum_{K_x^1}\sum_{K_y^1}u(K_x^1,K_y^1,t)v(K_x-K_x^1,K_y-K_y^1,t)\\+K_y\left[1-\frac{K_y^2}{K_x^2+K_y^2}\right]\sum_{K_x^1}\sum_{K_y^1}v(K_x^1,K_y^1,t)v(K_x-K_x^1,K_y-K_y^1,t)\end{array}\right\}$$

Continuity

$$u(K_x,K_y,t)\cdot K_x+v(K_x,K_y,t)\cdot K_y=0$$

Energy Spectrum

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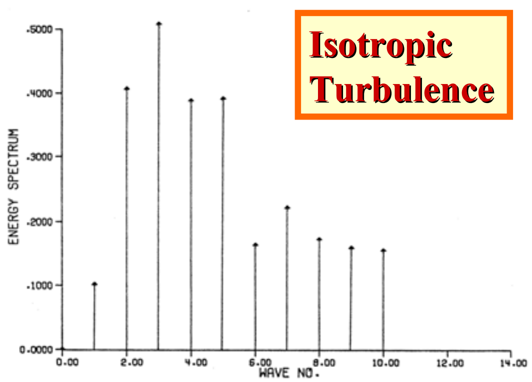

Isotropic Turbulence

Autocorrelations

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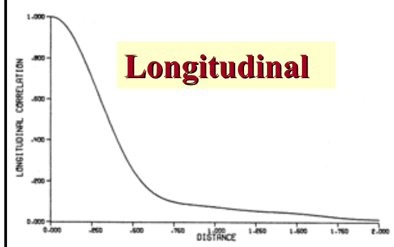
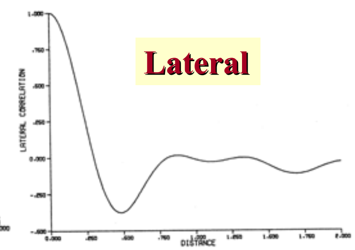
Isotropic Turbulence



Longitudinal



Lateral

Isotropic Turbulence

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Instantaneous Velocity Contours

Chaos and Turbulence

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Navier-Stokes

$$\frac{\partial \mathbf{u}}{\partial t} = N(\mathbf{u}, \mathrm{Re})$$

Equilibrium Solutions
- Time Invariant
- Time-Periodic
- Quasi-Periodic
- Chaotic

Chaos and Turbulence

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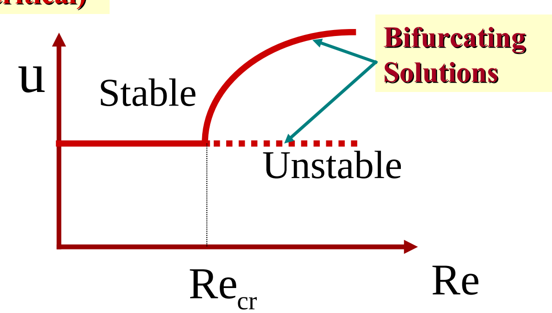
Bifurcation

(Supercritical)



Bifurcation

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Regular Bifurcation	Time Invariant	⇒	Time Invariant
Hopf Bifurcation	Time Invariant	⇒	Time Periodic
Chaotic Bifurcation	Quasi-Periodic	⇒	Chaotic

Chaos and Turbulence

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Landau-Hopf

Countably Infinite Bifurcation of Navier-Stokes Equation

Turbulence

Ruelle-Takens

After Four Bifurcation Solutions to Navier-Stokes Equation Become Chaotic

Turbulence

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G.I. Taylor & von Karman (1937)

"Turbulence is an irregular motion which in general makes its appearance in fluids, gaseous or liquid, when they flow past solid surfaces or even when neighboring streams of the same fluid flow past or over one another."

Turbulence

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Hinze (1959)

"Turbulent fluid motion is an irregular condition of flow in which the various quantities show a random variation with time and space coordinates, so that statistically distinct average values can be discerned."

Chaos - Turbulence

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Concluding Remarks

- Double Diffusive Convection
- Thermal Convection
- Isotropic Turbulence
- Bifurcation
- Turbulence