ME 639 - Turbulence

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DYNAMICAL SYSTEMS

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Dynamical Systems

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Outline

- Definitions
- Phase Plane
- Attractor and Stability
- Bifurcation

Dynamical Systems

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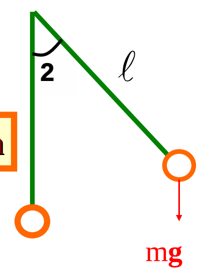
Engineering Systems ⟷ Dynamical Systems

Simple Pendulum



State ⟹ $\theta, \dot{\theta}$

Phase Plane ⟹ Plot of $\dot{\theta}$ versus θ

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Equation of Motion ⟷ System of Differential Equations $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, t)$

Simple Pendulum ⟹

$$\begin{cases} \dot{\theta} = \omega \\ \dot{\omega} = -2\zeta\omega_o\omega - \omega_o^2 \sin\theta \end{cases}$$

$$\omega_o^2 = \frac{g}{\ell}$$

Linear Pendulum ⟹

$$\begin{cases} \dot{\theta} = \omega \\ \dot{\omega} = -2\zeta\omega_o\omega - \omega_o^2\theta \end{cases}$$

Dynamical Systems, Definitions

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- **Autonomous Systems** ⟷ **Time Does Not Appear Explicitly**
- **Nonautonomous Systems** ⟷ **Time Appears Explicitly**
- **Orbit/ Trajectory** ⟷ **Curves in Phase Space Solutions of Equation of Motion**

Dynamical Systems, Definitions

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- **Flow** ⇒ **Vectors Tangent to Trajectories in Phase Space**
- **Null Cline** ⇒ **Lines in Phase Space for $\dot{x}_i = 0$**

Pendulum

$$\begin{cases} \dot{\theta} = \omega = 0 \\ \dot{\omega} = -\omega_o \sin\theta / 2\zeta \end{cases}$$

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- **Intersection of Clines** ⇒ **Equilibrium Points**

Poincare Section Poincare Map

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Non-Wandering Set

A set of points that orbits starting form this set come arbitrary close and arbitrary often to any point in the set

- Fixed (stationary) points
- Limit cycle (periodic)
- Quasi periodic
- Chaotic (bounded, non-periodic)

Stability and Bifurcation

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Nonlinear Dynamical Systems
- Qualitative Behavior
- Non-wandering sets
- Stability of non-wandering sets
- Changes in the number of non-wandering sets

Bifurcation

Appearance and Disappearance of Non-wandering sets

Stability

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Lyapunov (Marginal) Stability

A non-wandering set (NWS) is Lyapunov stable if every orbit starting in its neighborhood stays in its neighborhood.

Asymptotic Stability

A NWS is asymptotically stable if in addition to Lyapunov stability every orbit in its neighborhood approaches the NWS.

Attractors and Basin of Attraction

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Attractors

Asymptotically stable non-wandering sets are called attractors.

Basin of Attraction

The set of all initial states that approach the attractor.

Linear Stability

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Given an Orbit ⇒ $\mathbf{x}_o(t)$ $\qquad \dot{\mathbf{x}}_o = \mathbf{f}(\mathbf{x}_o, t)$

$\mathbf{x}_o(t)$ is asymptotically stable if $\Delta\mathbf{x}(t)$ decays ⇔ $\frac{d(\Delta\mathbf{x})}{dt} = \frac{d\mathbf{f}(\mathbf{x}_o)}{d\mathbf{x}} \cdot \Delta\mathbf{x}$

$$\mathrm{del}\left|\frac{d\mathbf{f}}{d\mathbf{x}} - \lambda\mathbf{I}\right| \Rightarrow \lambda_s$$

Solution ⇒ $\Delta\mathbf{x} = e^{\lambda_s t}\Delta\mathbf{x}_s$

Linear Stability

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Stable Orbit if all λ_s are negative

Routh-Hurwitz Theorem For two-D all the roots are negative if

$$\text{del}\left|\frac{\mathrm{d}\mathbf{f}}{\mathrm{d}\mathbf{x}}\right| > 0$$

$$\text{tr}\left|\frac{\mathrm{d}\mathbf{f}}{\mathrm{d}\mathbf{x}}\right| < 0$$

Nodes, Spiral and Saddle, 2-D

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- **Two real eigenvalues of the same sign** → **Nodes**
- **Pair of complex conjugate eigenvalues** → **Spirals**
- **Two real eigenvalues with opposite sign** → **Saddle**

Bifurcation

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A change in the number of attractors of a nonlinear dynamical systems with the change of a system parameter is called bifurcation.

Bifurcation is associated with the change of stability of an attractor.

In a bifurcation point, at least one eigenvalue of the Jacobian will attain a zero real part.

Stationary Bifurcation

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In a stationary bifurcation, a single real eigenvalue crosses the boundary of stability.

Hopf Bifurcation

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Hopf bifurcation occurs when a conjugated complex pair crosses the boundary of stability.

Pitchfork Bifurcation

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Consider a dynamical system → $\dot{u} = (\mu - u^2)u$

For $\mu<0$, $u=0$ is a stable equilibrium solution.

For $\mu>0$, $u=0$ is an unstable equilibrium solution, and $u= \pm\mu^{1/2}$ are stable solutions.

At $\mu=0$, a suppercritical Pitchfork bifurcation occurs.

Pitchfork Bifurcation

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Consider a dynamical system → $\dot{u} = (\mu + u^2)u$

For $\mu<0$, $u=0$ is a stable equilibrium solution, and $u= \pm(-\mu)^{1/2}$ are unstable solutions.

For $\mu>0$, $u=0$ is an unstable equilibrium solution.

At $\mu=0$, a subcritical Pitchfork bifurcation occurs.

Transcritical Bifurcation

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Consider a dynamical system → $\dot{u} = (\mu - u)u$

For $\mu<0$, $u=0$ is a stable equilibrium solution, and $u=\mu$ is an unstable solution.

For $\mu>0$, $u=0$ is an unstable solution, and $u=\mu$ is a stable equilibrium solution.

At $\mu=0$, the two solution exchange stability and a transcritical bifurcation occurs.