Dynamical Systems

Many engineering and natural systems are dynamical systems. For example a pendulum is a dynamical system.

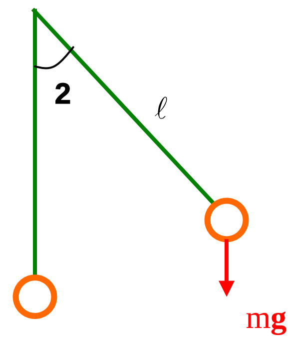


Figure 1. Simple pendulum.

State

The state of the dynamical system specifies it conditions. For a pendulum in the absence of external excitation shown in the figure, the angle θ and the angular velocity $\dot{\theta}$ uniquely define the state of the dynamical system.

Phase Space

Plots of the state variables against one another are referred to as the phase space representation. Every point in the phase space identifies a unique state of the system. For the pendulum, a plot of $\dot{\theta}$ versus θ is the phase space representation.

Equation of Motion

The equation of motion of a dynamical system is given by a set of differential equations. That is

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, t) \tag{1}$$

where **x** is the state and t is time. The dynamical system is linear if the governing equation is linear. For the pendulum shown in Figure 1, the equation of motion is given as

$$\begin{cases} \dot{\theta} = \omega \\ \dot{\omega} = -2\zeta\omega_o\omega - \omega_o^2 \sin\theta \end{cases} \qquad \text{where} \qquad \omega_o^2 = \frac{g}{\ell} \tag{2}$$

and the dynamical system is nonlinear. For small amplitude oscillation, $\sin\theta \approx \theta$, and the equation of motion becomes

$$\begin{cases} \dot{\theta} = \omega \\ \dot{\omega} = -2\zeta\omega_o\omega - \omega_o^2\theta \end{cases} \tag{3}$$

The dynamical system is now linear. In Equations (2) and (3) ω_o is the natural frequency and ζ is the damping coefficient.

Autonomous and Nonautonomous Systems

A system is said to autonomous if time does not appear explicitly in the equation of motion. The equation of motion of nonautonomous systems, however, explicitly depends on time. Thus, the equations of motion given by (2) and (3) for a pendulum in absence of external excitation are for autonomous systems, while a forced pendulum is a nonautonomous system. A nonautonomous system could be deterministic or stochastic.

Orbit

An orbit or trajectory is a curve in phase space, which is obtained by the solution of the equation of motion.

Flow

For a fixed time, **f(x,t),** the right-hand side of the equation of motion given by (1), identifies a vector field in the phase space that is tangent to the trajectories.

Figure 2 shows the phase space and the flow for a damped pendulum given by Equation (2). The black arrows of the vector field $\mathbf{f} = (\omega, -2\zeta\omega_o\omega - \omega_o^2 \sin\theta)$ are tangential to all trajectories in the phase space. A two-dimensional phase space provides a qualitative picture of the behavior of the dynamical systems including its orbit and its flow.

In Figure 2 the dashed red lines are null clines. These are the lines where the time derivative of one component of the state variable is zero. For the damped pendulum it follows that

$$\begin{cases} \dot{\theta} = \omega = 0 \\ \omega = -\omega_o \sin\theta / 2\zeta \end{cases} \tag{3}$$

Therefore, one null cline is the angle θ axes and the other is a sinusoidal curve in the phase space. On the first null cline the flow vector field is vertical (since $\dot{\theta} = 0$), while on the second one, the vector field is horizontal (since $\dot{\omega} = 0$).

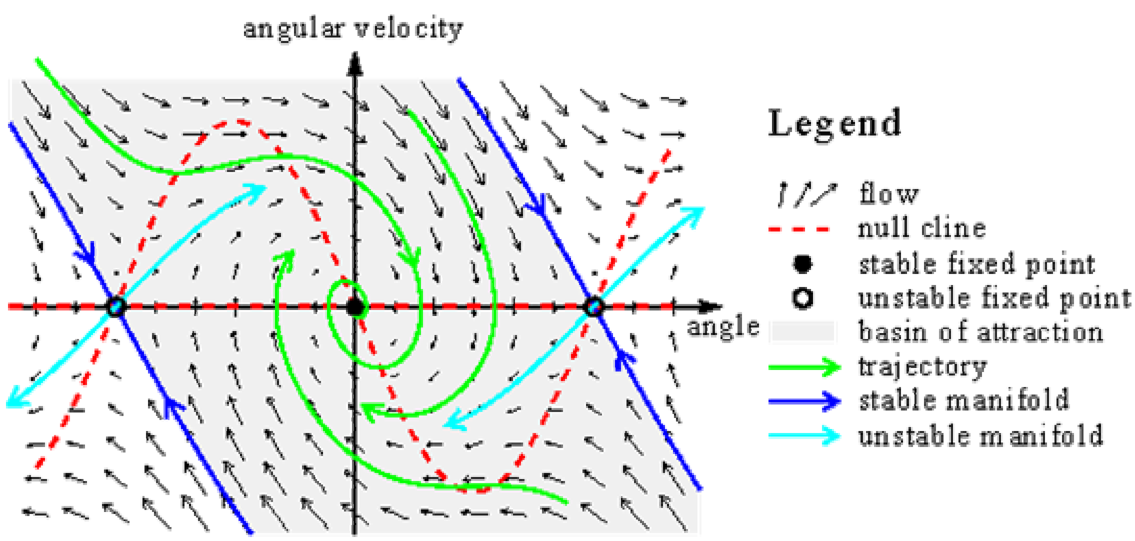


Figure 2. Schematics of the phase plane for the damped pendulum.
(http://monet.physik.unibas.ch/~elmer/pendulum/bterm.htm).

Between the null clines the direction of the flow vector is determined by the signs of $\dot{\theta}$ and $\dot{\omega}$. At the intersection points of the null clines, $\dot{\theta}=\dot{\omega}=0$, and the flow vector field is zero. These intersection points, which correspond to the stationary solutions, are called fixed points or equilibrium points. Fixed points of a dynamical system could be stable or unstable.

Poincaré Section and Poincaré Map

Poincaré section and Poincaré map are tools for visualization of the flow in a phase space. Poincaré section is a plane (or curved surface) in the phase space that is crossed by almost all orbits. The Poincaré map maps the points of the Poincaré section onto itself. Consecutive intersection points of the orbit with the Poincaré section form the Poincaré map. A Poincaré map represents a continuous dynamical system by a discrete one. Poincaré maps are invertable maps because one gets u_n from u_{n+1} by following the orbit backwards. For the case of the forced pendulum the Poincaré section is defined by a certain phase of the time-periodic excitation.

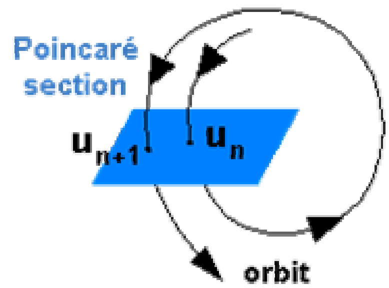


Figure 3. Schematics of Poincaré section and Poincaré map.

Non-Wandering Set

Non-wandering set is a set of points in the phase space for which all orbits starting from a point of this set come arbitrarily close and arbitrarily often to any point of the set. Non-wandering sets are of four types. These are

- Fixed (stationary) points. For the simple pendulum given by Equation (2), $\omega=\dot{\theta}=0$ and $\theta=0$ and $\theta=180°$ are fixed points.
- Limit cycles (periodic solutions). These solutions are common for the linearized oscillation of a simple pendulum.
- Quasi-periodic orbits. Periodic solutions with at least two incommensurable frequencies. These solutions occur for an undamped pendulum under periodic excitations.
- Chaotic orbits. Bound non-periodic solutions. These solutions occur for a (nonlinear) pendulum under certain external excitations.

The first three types can also occur in linear dynamical systems. The fourth type appears only in nonlinear systems. The Poincaré map for limit cycles become fixed points. A non-wandering set can be either stable or unstable. Changing a parameter of the system can change the stability of a non-wandering set. This is accompanied by a change of the number of non-wandering sets due to a bifurcation.

Stability and Bifurcation

In nonlinear dynamical system, the main questions are:

- What is the qualitative behavior of the dynamical system?
- How many non-wandering sets (fixed points, limit cycles, quasi-periodic or chaotic orbits) occur?
- Which of the non-wandering sets are stable?
- How does the number of non-wandering sets change with changes in the parameters of the system?

The appearance and disappearance of non-wandering sets is called a bifurcation. Change of stability and bifurcation always coincide.

Stability

A non-wandering set may be stable or unstable. The stability could be in the sense of Lyapunov (weak) or Asymptotic (strong).

Lyapunov (Marginal) Stability

A non-wandering set is said to be Lyapunov stable if every orbit starting in its neighborhood remains in its neighborhood.

Asymptotic Stability

A non-wandering set is said to be asymptotically stable if in addition to the Lyapunov stability, every orbit in its neighborhood approaches the non-wandering set asymptotically.

Thus, a non-wandering set is either asymptotically stable, marginally stable (Lyapunov stable), or unstable.

Attractors and Basin of Attraction

Asymptotically stable non-wandering sets are also called *attractors*. The basin of attraction is the set of all initial states approaching the attractor in the long time limit.

Linear Stability of Non-Wandering Set

To check the stability of a non-wandering set a linear stability analysis may be used. Let $\mathbf{x_o}(t)$ be an orbit that satisfies Equation (1). That is

$$\dot{\mathbf{x}}_o = \mathbf{f}(\mathbf{x}_o, t) \tag{4}$$

The solution $\mathbf{x_o}(t)$ is asymptotically stable if any infinitesimal small perturbation $\Delta\mathbf{x}(t)$ decays. Assume that $\mathbf{x} = \mathbf{x_o}(t) + \Delta\mathbf{x}$ is the perturbed solution, and $\mathbf{x}$ satisfies the equation of motion given by (1). It then follows that

$$\frac{d(\Delta\mathbf{x})}{dt} = \frac{d\mathbf{f}(\mathbf{x_o})}{d\mathbf{x}} \cdot \Delta\mathbf{x} \tag{5}$$

The linearized equation of motion for $\Delta\mathbf{x}(t)$ given by (5) is justified as long as the orbit is in the neighborhood of $\mathbf{x}_o$.

For fixed points, the fundamental solutions for $\Delta\mathbf{x}(t)$ are

$$\Delta\mathbf{x} = e^{\lambda_s t}\Delta\mathbf{x}_s \tag{6}$$

where λ_s and $\Delta\mathbf{x}_s$ are the eigenvalues and eigenvector of the Jacobian $\frac{d\mathbf{f}}{d\mathbf{x}}$, and s is the dimension of phase space. The eigenvalues λ_s are the roots of the characteristic polynomial

$$\mathrm{del}\left|\frac{d\mathbf{f}}{d\mathbf{x}} - \lambda\mathbf{I}\right| = 0 \tag{6}$$

The fixed point $\mathbf{x}_o$ is asymptotically stable if the real parts of all eigenvalues λ_s are negative. If at least one eigenvalue has a positive real part the corresponding fundamental solution would increase exponentially, and the fixed point will be unstable.

Routh and Hurwitz theorem may be used to check the stability without explicitely calculating the eigenvalues. For the case of a two-dimensional phase space the characteristic polynomial is quadratic. Routh and Hurwitz theorem implies that both eigenvalues have negative real part if and only if $\det\left|\frac{df_i}{dx_j}\right|$ is positive and the trace $\frac{df_i}{dx_i}$ is negative. The region of stability and instability are shown in Figure 4.

Nodes, Spiral and Saddle

Figure 4 shows a classification scheme of the fixed (stationary) points in two-dimensional phase spaces. The notion spiral and node are inspired by the flow near the fixed point. A pair of conjugated complex eigenvalues lead to a spiral, whereas a node is causes by two real eigenvalues of the same sign. Real eigenvalues of different sign lead to a saddle. That is, a saddle is a fixed point where at least one eigenvalue has a positive real part but also at least one eigenvalue has a negative real part. Near a saddle, an orbit is usually attracted at first but repelled later on. There are points in the phase space that approach the fixed point as $t \to \infty$, and they form the stable manifold. Saddles and their stable manifolds are usually the boundaries of a basin of attraction. The pendulum shown in Figure 1 has a saddle point (corresponding to the upside-down equilibrium position) and a spiral if $\zeta < 1$ or a node if $\zeta > 1$ as shown in Figure 2.

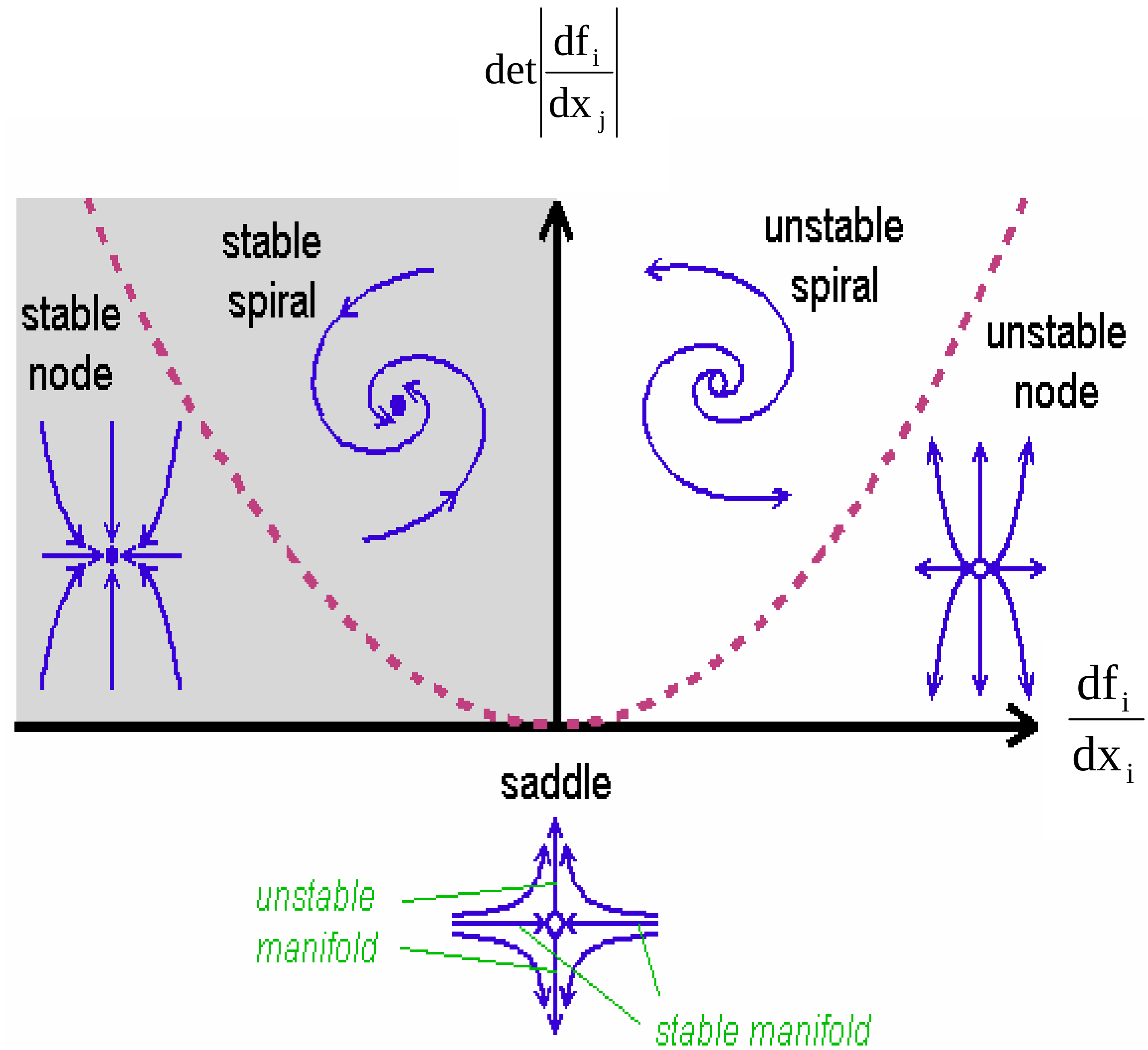


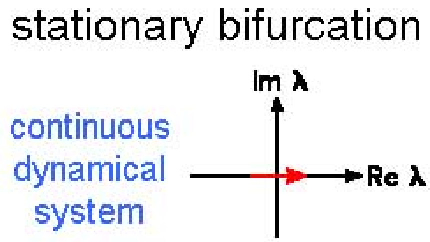
Figure 4. The region of stability and instability and classification scheme of the fixed (stationary) points in two-dimensional phase spaces.
http://monet.physik.unibas.ch/~elmer/pendulum/bif.htm

Bifurcation

The number of attractors in a nonlinear dynamical system can change when a system parameter is varied. This change is called bifurcation and it is accompanied by a change of the stability of an attractor. In a bifurcation point, at least one eigenvalue of the Jacobian will attain a zero real part. There are three generic types of bifurcation.

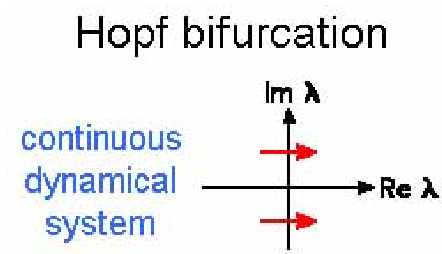
Stationary Bifurcation

In a stationary bifurcation, a single real eigenvalue crosses the boundary of stability.



Hopf Bifurcation

Hopf bifurcation occurs when a conjugated complex pair crosses the boundary of stability. In the time-continuous case, a limit cycle bifurcates. It has an angular frequency that is given by the imaginary part of the crossing pair.



In this section the main forms one-dimensional bifurcation are illustrated. The phase space variable is u. The control parameter is μ. The bifurcation point is at $\mu = 0$. The direction of motion in the one-dimensional phase space is shown by arrows. Stable (unstable) fixed points are drawn as red solid (dotted) lines.

Pitchfork Bifurcation

Pitchfork bifurcation is possible in dynamical systems with an inversion or reflection symmetry. That is, an equation of motion that remains unchanged if one changes the sign of all phase space variables. An example is a system governed by

$$\dot{u} = (\mu - u^2)u \tag{7}$$

As the control parameter μ varies the stationary solutions and their stability conditions changes. For $\mu < 0$, there is one equilibrium solution, u=0, which is stable. For $\mu > 0$, there are three equilibrium solutions u=0, and $u = \pm\sqrt{\mu}$. Here, u=0 is unstable while $u = \pm\sqrt{\mu}$ are stable. At $\mu = 0$, a bifurcation occurs which is referred to as (supercritical) pitchfork bifurcation.

Similarly for a system governed by

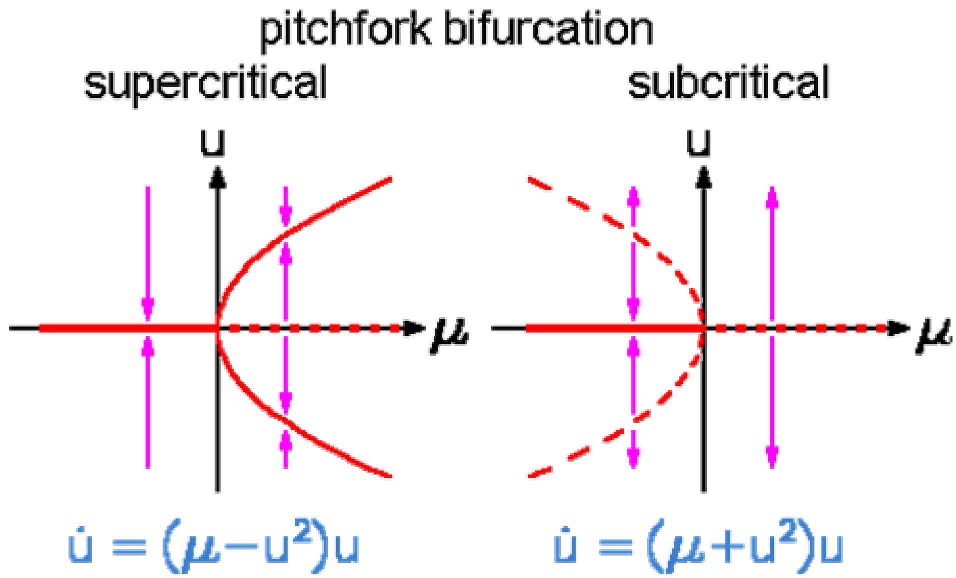


Figure 5. Schematics of a pitchfork bifurcation.

$$\dot{u} = (\mu + u^2)u \tag{8}$$

there are three equilibrium solutions u=0, and $u = \pm\sqrt{-\mu}$ for $\mu < 0$. In this case u=0 is a stable solution while $u = \pm\sqrt{-\mu}$ are unstable. For $\mu > 0$, there is only one solution, u=0, which is stable. In this case at $\mu = 0$, a (subercritical) pitchfork bifurcation occurs.

Transcritical Bifurcation

Consider a dynamical system given by

$$\dot{u} = (\mu - u)u \tag{9}$$

For $\mu < 0$, there are two equilibrium solution, u=0, which is stable, and $u = \mu$, which is unstable. For $\mu > 0$, there are still the same two equilibrium solution. But u=0 is unstable, and $u = \mu$ is now stable. That is at $\mu = 0$, the two stationary solution exchanged their stability. In this case it is said that a transcritical bifurcation occurred.