NONLINEAR THEORY OF STABILITY OF VISCOUS FLOW

Goodarz Ahmadi
Department of Mechanical and Aeronautical Engineering
Clarkson University
Potsdam, NY 13699-5725

STABILITY OF VISCOUS FLOW

Outline

- Nonlinear Stability Analysis
- Disturbed Motion
- Energy Method
- Uniqueness Theorems

Nonlinear Stability Analysis

Basic Motion satisfies the Navier-Stokes Equation:

$$\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v}\cdot\nabla\mathbf{v} = -\nabla p + \frac{1}{Re}\nabla^2\mathbf{v} \quad \text{in } V$$

$$\nabla\cdot\mathbf{v} = 0$$

Boundary Conditions

$$\mathbf{v}(\mathbf{x}) = \mathbf{V} \quad \text{on } S$$

Nonlinear Stability Analysis

Disturbed Motion $\mathbf{v}^*,\ p^*$

$$\frac{\partial \mathbf{v}^*}{\partial t} + \mathbf{v}^*\cdot\nabla\mathbf{v}^* = -\nabla p^* + \frac{1}{Re}\nabla^2\mathbf{v}^*$$

$$\nabla\cdot\mathbf{v}^* = 0$$

Boundary Conditions

$$\mathbf{v}^* = \mathbf{V} \quad \text{on } S$$

Difference Motion

Clarkson University

$$\mathbf{u} = \mathbf{v}^* - \mathbf{v}$$

$$\pi = p^* - p$$

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u}\cdot\nabla\mathbf{v} + \mathbf{v}\cdot\nabla\mathbf{u} + \mathbf{u}\cdot\nabla\mathbf{u} = -\nabla\pi + \frac{1}{Re}\nabla^2\mathbf{u}$$

Boundary Conditions → $\mathbf{u} = \mathbf{0} \quad \text{on} \quad S$

Energy Stability Analysis → $T = \frac{1}{2}\int u^2 dV$

Energy Stability Analysis

Clarkson University

$$\frac{dT}{dt} = \int \mathbf{u}\cdot\frac{\partial \mathbf{u}}{\partial t} dV = \int\left[\frac{1}{Re}\mathbf{u}\cdot\nabla^2\mathbf{u} - \mathbf{u}\cdot\nabla\pi - \mathbf{u}\cdot\nabla\mathbf{v}\cdot\mathbf{u} - \mathbf{v}\cdot\nabla\mathbf{u}\cdot\mathbf{u} - \mathbf{u}\cdot\nabla\mathbf{u}\cdot\mathbf{u}\right] dV$$

Vector Identities

$$\nabla\times\nabla\times\mathbf{u} = \nabla(\nabla\cdot\mathbf{u}) - \nabla^2\mathbf{u}$$

$$\mathbf{u}\cdot\nabla\times(\nabla\times\mathbf{u}) = -\nabla\cdot(\mathbf{u}\times(\nabla\times\mathbf{u})) + (\nabla\times\mathbf{u})^2$$

Energy Stability Analysis

Clarkson University

$$\int \mathbf{u}\cdot\nabla^2\mathbf{u}\,dV = \int_S \mathbf{u}\times(\nabla\times\mathbf{u})\cdot d\mathbf{S} - \int(\nabla\times\mathbf{u})^2 dV$$

$$\int \mathbf{u}\cdot\nabla\pi\,dV = \int[\nabla\cdot(\pi\mathbf{u}) - \pi\nabla\cdot\mathbf{u}]dV = \int_S \pi\mathbf{u}\cdot d\mathbf{S} = 0$$

General Energy Stability Equation

$$\frac{dT}{dt} = -\frac{1}{Re}\int(\nabla\times\mathbf{u})^2 dV - \int \mathbf{u}\cdot\nabla\mathbf{v}\cdot\mathbf{u}\,dV$$

Energy Stability Analysis

Clarkson University

Korn Inequality

$$\int(\nabla\times\mathbf{u})^2 dV \geq N\int u^2 dV$$

$$-\mathbf{u}\cdot\mathbf{d}\cdot\mathbf{u} \leq \lambda u^2$$

General Energy Stability Equation

$$\frac{dT}{dt} \leq 2\left(-\frac{N}{Re} + \lambda\right)T$$

Energy Stability Analysis

Clarkson University

$$\mathrm{Re} \le \frac{N}{\lambda} \quad \Rightarrow \quad \textbf{Stability}$$

$$T \le T(0)\exp\left\{-\left(\frac{N}{\mathrm{Re}} - \lambda\right)t\right\}$$

$t \to \infty$ $T \to 0$ $u = 0$ $\mathbf{v}^* = \mathbf{v}$

Energy Stability Analysis Theorem

Clarkson University

If for a basic flow of a viscous incompressible fluid in a bounded region of space

$$\mathrm{Re} \le \frac{N}{\lambda}$$

then the basic flow is stable.

Corollary 1 (Uniqueness) (Unsteady Viscous Flows)

Clarkson University

If v and v^* are two unsteady flows of a viscous fluid in a bounded region of space having the same velocity distribution at time t=0 and on surface boundary S, then they must be identical if

$$\mathrm{Re} \le \frac{N}{\lambda}$$

Corollary 2 (Uniqueness) (Steady Viscous Flows)

Clarkson University

If v and v^* are two steady flows of a viscous fluid in a bounded region of space V(t) subject to the same boundary conditions on surface boundary S, then they must be identical if

$$\mathrm{Re} \le \frac{N}{\lambda}$$