Stability of Viscous Flows

Consider the Navier-Stokes equation in dimensionless form

$$\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{v} = -\nabla p + \frac{1}{\mathrm{Re}} \nabla^2 \mathbf{v}, \tag{1}$$

where $\mathbf{v}$ and P are nondimensional velocity and pressure and Re is the Reynolds number. The continuity equation is given by

$$\nabla \cdot \mathbf{v} = 0. \tag{2}$$

Let $V(\mathbf{x})$, P be a steady basic flow, the stability of which is to be analyzed. That is,

$$\mathbf{V} \cdot \nabla \mathbf{V} = -\nabla \mathbf{P} + \frac{1}{\mathrm{Re}} \nabla^2 \mathbf{V}, \tag{3}$$

$$\nabla \cdot \mathbf{V} = 0 \tag{4}$$

Now let the disturbed motion be given as

$$\mathbf{v} = \mathbf{V} + \mathbf{v}', \qquad |v'| << |V| \tag{5}$$

$$p = P + p'. \tag{6}$$

Using (5) and (6) in (1), and subtracting (3) we find

$$\frac{\partial \mathbf{v}'}{\partial t} + \mathbf{V} \cdot \nabla \mathbf{v}' + \mathbf{v}' \cdot \nabla \mathbf{V} = -\nabla p' + \frac{1}{\mathrm{Re}} \nabla^2 \mathbf{v}', \tag{7}$$

$$\nabla \cdot \mathbf{v}' = 0. \tag{8}$$

In the derivation of equation (7), we neglected $\mathbf{v}' \cdot \nabla \mathbf{v}'$ which is of higher order infinitesimal.

Stability of Two-Dimensional Parallel Flows

Consider the special case where

$$\mathbf{V} = U(y)\mathbf{i}, \qquad P = P(x, y). \tag{9}$$

Equations (7) and (8) for two-dimensional flows may be restated as

$$\frac{\partial u'}{\partial t} + U\frac{\partial u'}{\partial x} + v'\frac{\partial U}{\partial y} = -\frac{\partial P'}{\partial x} + \frac{1}{Re}\nabla^2 u', \tag{10}$$

$$\frac{\partial v'}{\partial t} + U\frac{\partial v'}{\partial x} = -\frac{\partial P'}{\partial y} + \frac{1}{Re}\nabla^2 u', \tag{11}$$

$$\frac{\partial u'}{\partial x} + \frac{\partial v'}{\partial y} = 0. \tag{12}$$

Introducing the stream function ψ for the disturb motion with

$$u' = \frac{\partial \psi}{\partial y}, \qquad v' = -\frac{\partial \psi}{\partial x}, \tag{13}$$

and eliminating p' between (10) and (11), the results may be restated as

$$\frac{\partial}{\partial t}\nabla^2\psi + U\frac{\partial}{\partial x}\nabla^2\psi - \frac{\partial \psi}{\partial x}U'' = \frac{1}{Re}\nabla^4\psi. \tag{14}$$

Equation (14) governs the dynamic of the disturbed motion.

Now assume a propagating wave solution given as

$$\psi = \varphi(y)e^{i\alpha(x-ct)}, \tag{15}$$

where α is the wave number and c is the complex speed of the wave. i.e.,

$$c = c_r + ic_i \tag{16}$$

and c_i is the important parameter for stability analysis. That is, the disturbance is damped if c_i is negative. For $c_i > 0$, the disturbance will grow and leads to instability. Using (15) in (14), we find the Orr-Sommerfeld equation. i.e.,

$$(U-c)(\varphi'' - \alpha^2\varphi) - U''\varphi = -\frac{i}{\alpha Re}(\varphi'''' - 2\alpha^2\varphi'' + \alpha^4\varphi) \tag{17}$$

Equation (17) is the basis for linear stability analysis of parallel viscous flows.

The boundary conditions for a boundary layer type flow are:

At $y = 0$, $u' = v' = 0$ or $\varphi(0) = \varphi'(0) = 0$

$$\text{At } y = \infty, \; u' = v' = 0 \text{ or } \varphi(\infty) = \varphi'(\infty) = 0 \tag{18}$$

Equation (17) together with boundary conditions given in (18) form a complex eigenvalue problem. For given α and $c_i = 0$ (neutral stability), the eigenvalues c_r and Re may be found. A typical curve for the case of boundary layer over a plate is shown in the figure.

In this figure, the critical Reynolds number is given by

$$\mathrm{Re}_{crit} = \left.\frac{U_\infty \delta^*}{\nu}\right|_{crit} = 520 \tag{19}$$

with δ^* being the displacement thickness. For $\mathrm{Re} > \mathrm{Re}_{crit}$, some modes become unstable. At the critical Reynolds number, $\alpha\delta^* \approx 0.35$. The critical wavelength becomes

$$\lambda_{crit} \approx 18\delta^* \approx 6\delta.$$

Here δ is the boundary layer thickness.

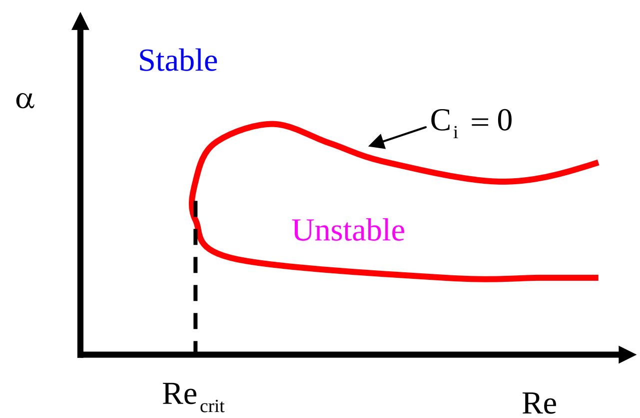

Figure 1. Stability boundary for boundary layer flows.

Experimental data shows that the critical value of Reynolds number as defined in Equation (19) is about 950 to 1700 (corresponding to $\mathrm{Re}_x = \frac{U_\infty x}{\nu} \approx 3.2 \times 10^5 \sim 10^6$) depending on the free stream turbulence. The reasons for this discrepancy are as follows:

i. Unstable waves need a distance to travel before amplifying to a detectable level.
ii. Nonlinear effects may alter the nature of stability criterion.

Squire Theorem: Two-dimensional disturbances are more critical in comparison to the three dimensional disturbances for two-dimensional flows.

Squire Theorem implies that for linear stability analysis of two-dimensional flows, we need to be only concerned with planar disturbances. When the flow is stable under two-dimensional disturbances, it will stay stable under three-dimensional disturbance as well,

Frictionless Stability Analysis

For the frictionless case, equation (17) becomes

$$(U - c)(\varphi'' - \alpha^2 \varphi) - U'' \varphi = 0 . \tag{19}$$

The associated boundary conditions are

$$\varphi(0) = \varphi(\infty) = 0 . \tag{20}$$

Stability Theorem (Rayleigh, Tollmien)

The boundary layer velocity profiles that possess a point of inflexion are unstable.

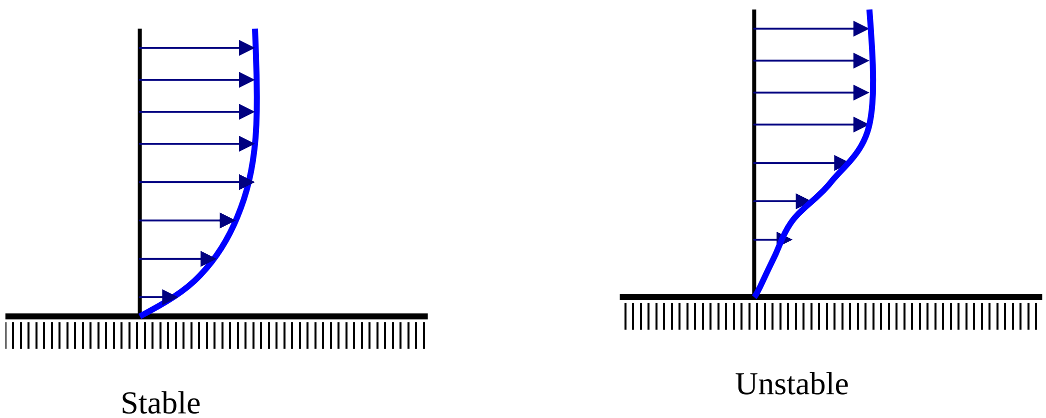

Figure 2. Schematics of boundary layer velocity profiles for stable and unstable flows.