CONSERVATION LAWS

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CONSERVATION LAWS

Outline

- Conservation Laws
- Constitutive Equations
- Navier-Stokes Equation
- Heat Transfer Equation
- Dimensionless Groups

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Axiom 1: Principle of Conservation of Mass

Mass is invariant under the motion

$$\frac{d}{dt}\int_v \rho dv = 0$$

Global

$$\frac{\partial}{\partial t}\int_v \rho dv + \int_s \rho \mathbf{v}\cdot \mathbf{ds} = 0$$

Local

$$\frac{\partial \rho}{\partial t} + (\rho v_k)_{,k} = 0$$

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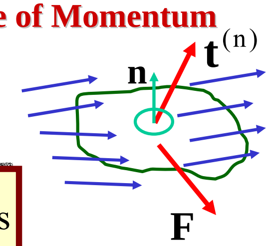
Axiom 2: Principle of Balance of Momentum

$$\frac{d}{dt}(\text{Momentum}) = \sum \text{Forces}$$



Global

$$\frac{d}{dt}\int_v \rho v_k dv = \int_v \rho f_k dv + \int_s t_k^{(n)} ds$$

Stress Tensor

$$\mathbf{t}^{(n)} = \mathbf{n}\cdot\mathbf{t}$$

Divergence Theorem

$$\int_s t_{\ell k} n_\ell ds = \int_v t_{\ell k,\ell} dv$$

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Axiom 2: Principle of Balance of Momentum

$$\int_v \left(\rho \frac{dv_k}{dt} - \rho f_k - t_{\ell k,\ell}\right) dv = 0$$

Local

$$\rho \frac{dv_k}{dt} = \rho f_k + t_{\ell k,\ell}$$

$$\rho \frac{d\mathbf{v}}{dt} = \rho \mathbf{f} + \nabla \cdot \mathbf{t}$$

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Axiom 3: Principle of Balance of Angular Momentum

Global

$$\underbrace{\frac{d}{dt}\int_v \rho(\sigma_k + \varepsilon_{kmj} r_m v_j)\,dv}_{\text{Time rate of change of angular momentum}} = \underbrace{\int_v \rho \varepsilon_{kmj} r_m f_j\,dv}_{\text{Moment of body force}} + \underbrace{\int_S \varepsilon_{kmj} r_m t_j^{(n)}\,ds}_{\text{Moment of surface force}} + \int_S \underbrace{m_k^{(n)}\,ds}_{\text{Couple Stress}} + \underbrace{\int_S \rho \ell_k\,ds}_{\text{Body couple}}$$

Axiom 3: Principle of Balance of Angular Momentum

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Local

$$\rho \dot{\sigma}_k = \rho \ell_k + \varepsilon_{kmj} t_{mj} + m_{\ell k,\ell}$$

When $\sigma_k = \ell_k = m_{k\ell} = 0$ $\quad \varepsilon_{kmj} t_{mj} = 0$

$$t_{mj} = t_{jm}$$

Stress Tensor is Symmetric

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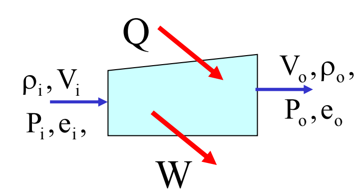
Axiom 4: Principle of Conservation of Energy



$$\underbrace{\frac{d}{dt}(K+E)}_{\text{Time rate of change kinetic and internal enrgies}} = \underbrace{W}_{\text{Work done by all the forces}} + \underbrace{Q}_{\text{Heat transferred}}$$

Axiom 4: Principle of Conservation of Energy

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Global

$$\frac{d}{dt}\int_v \rho\left(e+\frac{1}{2}v_k v_k\right)dv = \int_v \rho v_k f_k dv + \int_s v_k \cdot t_k^{(n)} ds + \int_s q_k ds_k + \int_v \rho r dv$$

Axiom 4: Principle of Conservation of Energy

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Local

$$\rho\dot{e} = t_{\ell k} v_{\ell,k} + q_{k,k} + \rho r$$

$\mathbf{q}$ = heat flux

r = heat source

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Axiom 5: Entropy Inequality

Global

$$\underbrace{\frac{d}{dt}\int_v \rho\eta dv}_{\text{Time rate of change of entropy}} \underbrace{- \int_S \frac{q_k n_k}{T} ds - \int_v \frac{\rho r}{T} dv}_{\text{Heat transfer divided by temperature}} \geq 0$$

Local

$$\rho\dot{\eta} - \left(\frac{q_k}{T}\right)_{,k} - \frac{\rho r}{T} \geq 0$$

Summary of Conservation Laws - Local

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Mass

$$\frac{\partial \rho}{\partial t} + \nabla\cdot(\rho\mathbf{v}) = 0$$

Momentum

$$\rho\frac{d\mathbf{v}}{dt} = \rho\mathbf{f} + \nabla\cdot\mathbf{t}$$

$$\mathbf{t} = \mathbf{t}^T$$

Energy

$$\rho\frac{de}{dt} = \mathbf{t}:\nabla\mathbf{v} + \nabla\cdot\mathbf{q} + \rho r$$

Entropy

$$\rho\frac{d\eta}{dt} - \nabla\cdot\left(\frac{\mathbf{q}}{T}\right) - \frac{\rho r}{T} \geq 0$$

CONSTITUTIVE EQUATIONS

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Continuum Thermodynamics

Helmholtz Free Energy $$\psi = e - T\eta$$

Entropy $$\frac{\rho}{T}(\dot{e} - \dot{T}\eta - \dot{\psi}) - (\frac{q_k}{T})_{,k} - \frac{\rho r}{T} \geq 0$$

Clausius-Duhem $$\frac{1}{T}\left[-\rho(\dot{\psi} + \eta\dot{T}) + t_{\ell k} v_{\ell,k} + \frac{q_k T_{,k}}{T}\right] \geq 0$$

CONSTITUTIVE EQUATIONS

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Constitutive Postulates

Assuming $$\psi = \psi(T, \rho, d_{k\ell}, T_{,k})$$

$$\dot{\psi} = \frac{\partial \psi}{\partial T}\dot{T} + \frac{\partial \psi}{\partial \rho}\dot{\rho} + \frac{\partial \psi}{\partial d_{k\ell}}\dot{d}_{k\ell} + \frac{\partial \psi}{\partial T_{,k}}\dot{\overline{T_{,k}}}$$

Pressure $$p = -\frac{\partial \psi}{\partial \rho^{-1}} = \rho^2 \frac{\partial \psi}{\partial \rho}$$

$$\dot{\rho} = -\rho d_{kk} \qquad \frac{\partial \psi}{\partial \rho}\dot{\rho} = -\frac{p}{\rho} d_{kk}$$

CONSTITUTIVE EQUATIONS

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Entropy Equation

$$\frac{1}{T}\left[-\rho(\frac{\partial \psi}{\partial T} + \eta)\dot{T} + (t_{k\ell} + p\delta_{k\ell})d_{k\ell} - \rho\frac{\partial \psi}{\partial T_{,k}}\dot{\overline{T_{,k}}} - \rho\frac{\partial \psi}{\partial d_{k\ell}}\dot{d}_{k\ell} + \frac{q_k T_{,k}}{T}\right] \geq 0$$

$$\eta = -\frac{\partial \psi}{\partial T} \qquad \frac{\partial \psi}{\partial T_{,k}} = \frac{\partial \psi}{\partial d_{k\ell}} = 0$$

$$(t_{k\ell} + p\delta_{k\ell})d_{k\ell} + \frac{q_k T_{,k}}{T} \geq 0$$

CONSTITUTIVE EQUATIONS

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Linear Constitutive Equations

$$t_{k\ell} = -p\delta_{k\ell} + L_{k\ell ij} d_{ij}$$

$$q_k = L_{kj} T_{,j}$$

$$L_{k\ell ij} d_{ij} d_{k\ell} \geq 0 \qquad L_{kj} T_{,k} T_{,j} \geq 0$$

Linear Constitutive Equations

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Isotropic Materials

$$L_{k\ell ij} = \lambda \delta_{k\ell}\delta_{ij} + \mu(\delta_{ki}\delta_{\ell j} + \delta_{kj}\delta_{\ell i}) + \mu_1(\delta_{ki}\delta_{\ell j} - \delta_{kj}\delta_{\ell i})$$

$$L_{k\ell} = \kappa \delta_{k\ell}$$

Newtonian Fluids

$$t_{k\ell} = (-p + \lambda d_{ii})\delta_{k\ell} + 2\mu d_{k\ell}$$

Fourier Law

$$q_k = \kappa T_{,k}$$

Linear Constitutive Equations

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Thermodynamical Constraints

$$3\lambda + 2\mu \ge 0 \qquad \mu \ge 0 \qquad \kappa \ge 0$$

Stokes Assumption

$$\lambda = -\frac{2}{3}\mu \iff p = -\frac{1}{3}t_{kk}$$

$$t_{k\ell} = -p\delta_{k\ell} + 2\mu d^D_{k\ell} \qquad d^D_{ij} = d_{ij} - \frac{1}{3}d_{kk}\delta_{ij}$$

Navier-Stokes Equation

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$$\rho \frac{dv_k}{dt} = -p_{,k} + \mu v_{k,jj} + (\lambda + \mu) v_{j,jk} + \rho f_k$$

Incompressible Fluids

$$\nabla \cdot \mathbf{v} = 0$$

$$\rho \frac{d\mathbf{v}}{dt} = -\nabla p + \mu \nabla^2 \mathbf{v} + \rho \mathbf{f}$$

Navier-Stokes Equation

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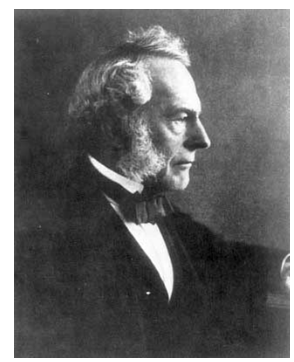
Navier



Stokes

Energy Equation

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$$\rho\frac{de}{dt}=\mathbf{t}:\nabla\mathbf{v}+\nabla\cdot\mathbf{q}+\rho r$$

$$q_k=\kappa T_{,k}$$

Heat Transfer

$$\rho\frac{de}{dt}=\nabla\cdot(\kappa\nabla T)+t_{ij}v_{j,i}+\rho r$$

$$t_{ij}v_{j,i}=-pv_{k,k}+\Phi$$

Dissipation

$$\Phi=\lambda v_{k,k}v_{i,i}+2\mu d_{ij}v_{j,i}$$

Energy Equation in term of Enthalpy

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$$pv_{k,k}=-\frac{p}{\rho}\frac{d\rho}{dt}=\rho\frac{d}{dt}\left(\frac{p}{\rho}\right)-\frac{dp}{dt}$$

Enthalpy

$$h=e+\frac{p}{\rho}$$

$$\rho\frac{dh}{dt}=\frac{dp}{dt}+\nabla\cdot(\kappa\nabla T)+\Phi+\rho r$$

Heat Capacities

$$dh=c_p dT \qquad de=c_v dT$$

Heat Transfer Equation

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$$\rho c_P\frac{dT}{dt}=\frac{dP}{dt}+\kappa\nabla^2 T+\Phi+\rho r$$

Incompressible Flow

$$\rho c_v\frac{dT}{dt}=\kappa\nabla^2 T+\Phi+\rho r$$

Dissipation

$$\Phi=\mu(v_{i,j}+v_{j,i})v_{j,i}$$

Boussinesq Approximation

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Thermal Expansion

$$\rho=\rho_0(1-\beta(T-T_0))$$

Body Force

$$\rho\mathbf{f}=-\rho_0 g\mathbf{k}[1-\beta(T-T_0)]$$

Boussinesq Equation

$$\rho_0\frac{d\mathbf{v}}{dt}=-\nabla\hat{P}+\mu\nabla^2\mathbf{v}-\rho_0 g\beta(T-T_0)\mathbf{k}$$

$$\hat{P}=p+\rho_0 gz$$

Dimensionless Equations

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Dimensionless Variables

$$x_i^* = \frac{x_i}{L} \quad \mathbf{v}^* = \frac{\mathbf{v}}{U_\infty} \quad t^* = \frac{tU_\infty}{L} \quad \rho^* = \frac{\rho}{\rho_0}$$

$$P^* = \frac{\hat{P} - P_\infty}{\rho_0 U_\infty^2} \quad T^* = \frac{T - T_0}{\Delta T_0} \quad \mathbf{f}^* = \frac{\mathbf{f}}{g}$$

Dimensionless Equations

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Mass

$$\frac{\partial \rho^*}{\partial t^*} + \nabla^* \cdot (\rho^* \mathbf{v}^*) = 0$$

Momentum

$$\rho^* \frac{d\mathbf{v}^*}{dt^*} = -\nabla^* P^* + \frac{1}{Re} \nabla^{*2} \mathbf{v}^* - \frac{Gr}{Re^2} T^* \mathbf{f}^*$$

Energy

$$\rho^* \frac{dT^*}{dt^*} = Ec \frac{dP^*}{dt^*} + \frac{1}{Re\,Pr} \nabla^{*2} T^* + \frac{Ec}{Re} \Phi^*$$

Dimensionless Groups

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Reynolds Number

$$Re = \frac{\rho_0 U_\infty L}{\mu}$$

Prandtl Number

$$Pr = \frac{\mu c_P}{\kappa}$$

Grashof Number

$$Gr = \frac{g \beta \rho_0^2 L^3 \Delta T_0}{\mu^2}$$

Eckert Number

$$Ec = \frac{U_\infty^2}{c_P \Delta T_0}$$

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Concluding Remarks

- Conservation Laws
- Constitutive Equations
- Navier-Stokes Equation
- Heat Transfer Equation
- Dimensionless Groups