

## INDICIAL NOTATION (Cartesian Tensor)

### Basic Rules

- i) A free index appears only once in each term of a tensor equation. The equation then holds for all possible values of that index.
- ii) Summation is implied on an index, which appears twice.
- iii) No index can appear more than twice in any term.

### Definition (Cartesian Tensors)

Consider a change of frame

$$x_i^* = Q_{ij}x_j, \quad x_j = Q_{ij}x_i^*, \quad \det|Q_{ij}| = \pm 1, \quad Q_{ij}Q_{ik} = \delta_{jk}, \quad Q_{ij}Q_{kj} = \delta_{ik}$$

The quantities  $T(\mathbf{x})$ ,  $\mathbf{v}(\mathbf{x})$ , and  $\mathbf{t}(\mathbf{x})$  are said to be a scalar, a vector, or a second order tensor if they transform according to the rules:

$$T^* = T, \quad \mathbf{v}^* = \mathbf{Q} \cdot \mathbf{v}, \quad \boldsymbol{\tau}^* = \mathbf{Q} \cdot \mathbf{t} \cdot \mathbf{Q}^T$$

or in component forms:

$$v_i^* = Q_{ij}v_j, \quad t_{ij}^* = Q_{ik}Q_{jl}t_{kl}$$

Similarly, a third order tensor  $\lambda$  transforms as:

$$\lambda_{ijk}^* = Q_{im}Q_{jn}Q_{kl}\lambda_{mnl}$$

### Tensor Operations

Gradient:

$$(\nabla\varphi)_i = \frac{\partial\varphi}{\partial x_i} = \varphi_{,i}$$

$$(\nabla\mathbf{v})_{ij} = \frac{\partial v_j}{\partial x_i} = v_{j,i}$$

Divergence:

$$\nabla \cdot \mathbf{v} = v_{i,i}$$

$$(\nabla \cdot \boldsymbol{\tau})_j = \frac{\partial \tau_{ij}}{\partial x_i} = \tau_{ij,i}$$

Curl:

$$(\nabla \times \mathbf{U})_i = \varepsilon_{ijk} \frac{\partial U_k}{\partial x_j} = \varepsilon_{ijk} U_{k,j}$$

Here,  $\varepsilon_{ijk}$  is the permutation symbol. The permutation symbol is used for evaluating the determinant. e.g.,

$$\det|\mathbf{A}| = \varepsilon_{ijk} A_{1i} A_{2j} A_{3k},$$

The inner product of the permutation symbol satisfy the following identity:

$$\varepsilon_{ijk} \varepsilon_{imn} = \delta_{jm} \delta_{kn} - \delta_{jn} \delta_{km}$$

Laplacian:

$$\nabla^2 \varphi = \frac{\partial^2 \varphi}{\partial x_i \partial x_i} = \varphi_{,ii}$$

### Isotropic Tensors

Isotropic tensors are tensors, which are form invariant under all possible rotations of the frame of reference. The most general forms of the isotropic tensors are:

Rank zero: All scalars isotropic

Rank one: None

Rank two:  $\alpha \delta_{ij}$ ,  $\alpha$  a scalar and  $\delta_{ij}$  = Kronecker delta

Rank three:  $\alpha \varepsilon_{ijk}$

Rank four:  $\alpha \delta_{ij} \delta_{kl} + \beta (\delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk}) + \gamma (\delta_{ik} \delta_{jl} - \delta_{il} \delta_{jk})$