Clarkson University

PHENOMENOLOGICAL MODELS FOR TURBULENCE

Reynolds Equation

Since turbulence is a continuum phenomenon, the instantaneous velocity and pressure fields satisfy the Navier-Stokes equation. i.e.,

$$\rho\left(\frac{\partial u_i}{\partial t}+u_j\frac{\partial u_i}{\partial x_j}\right)=-\frac{\partial p}{\partial x_i}+\mu\frac{\partial^2 u_i}{\partial x_j \partial x_j} \tag{1}$$

$$\frac{\partial u_i}{\partial x_i}=0. \tag{2}$$

During a turbulent motion $\mathbf{u}$ and p are random functions of space and time. Hence, these may be decomposed into mean and fluctuating parts, i.e.

$$\mathbf{u}=\mathbf{U}+\mathbf{u}',\ U_i=\overline{u_i},\ \overline{u_i'}=0, \tag{3}$$

$$p=P+p',\ P=\overline{p},\ \overline{p'}=0, \tag{4}$$

where $\mathbf{U}$ and P are the mean quantities and $\mathbf{u}'$ and p' are the fluctuating parts. Here, a bar on the top of the letter stands for the (time) averaged quantity. That is

$$\overline{u_i}=\lim_{T\to\infty}\frac{1}{T}\int_{t_0}^{t+T}u_i\,dt \tag{5}$$

The probabilistic (ensemble) average is defined as

$$<u_i>=\int_{-\infty}^{+\infty}u_i f(\mathbf{u})d\mathbf{u}, \tag{6}$$

where $f(\mathbf{u})$ is the probability density function of $\mathbf{u}$. Ergodicity assumption implies that the time average and ensemble average are equal. Hence,

$$\overline{u_i}=<u_i>=U_i \tag{7}$$

Note that the ergodicity hypothesis has not been proven for turbulence; however, it is commonly used to relate the theoretical results to the experimental data.

It is also well known that while $\overline{u_i'}=0$, $\overline{p'}=0$,

$$\overline{u_i' u_j'} \neq 0, \quad \overline{p' u_i'} \neq 0, \quad \overline{u_i' u_j' u_k'} \neq 0. \tag{8}$$

About a century ago, Reynolds suggested to use the decomposition given by (2) and (3) into the Navier-Stokes equation and average the resulting equation. Noting that

$$\overline{U_i u_j'} = U_i \overline{u_j'} = 0, \quad \frac{\partial \overline{u_i'}}{\partial x_j} = 0, \tag{9}$$

it follows that

$$\rho\left(\frac{\partial U_i}{\partial t} + U_j \frac{\partial U_i}{\partial x_j}\right) = -\frac{\partial P}{\partial x_i} + \mu \frac{\partial^2 U_i}{\partial x_j \partial x_j} - \rho \frac{\partial \overline{u_i' u_j'}}{\partial x_j}, \tag{10}$$

$$\frac{\partial U_i}{\partial x_i} = 0 \tag{11}$$

Equation (10) is referred to as the Reynolds equation. The term $\rho \overline{u_i' u_j'}$ is the stress induced by the turbulent fluctuation. i.e.,

$$\tau_{ij}^T = -\rho \overline{u_i' u_j'} = \tau_{ji}^T \tag{12}$$

Equation (10) may be restated as

$$\rho\left(\frac{\partial U_i}{\partial t} + U_j \frac{\partial U_i}{\partial x_j}\right) = \frac{\partial}{\partial x_j}\left[-P\delta_{ij} + \mu\left(\frac{\partial U_i}{\partial x_j} + \frac{\partial U_j}{\partial x_i}\right) - \rho \overline{u_i' u_j'}\right] \tag{13}$$

where $\tau_{ij} = -P\delta_{ij} + \mu\left(\frac{\partial U_i}{\partial x_j} + \frac{\partial U_j}{\partial x_i}\right)$ is the mean viscous stress tensor and $\rho \overline{u_i' u_j'}$ is the turbulent stress tensor.

The Reynolds stress (turbulent stress) is a symmetric tensor and its components are given by

$$\mathbf{t}^T = \begin{pmatrix} -\rho\overline{u'^2} & -\rho\overline{u'v'} & -\rho\overline{u'w'} \\ -\rho\overline{u'v'} & -\rho\overline{v'^2} & -\rho\overline{v'w'} \\ -\rho\overline{u'w'} & -\rho\overline{v'w'} & -\rho\overline{w'^2} \end{pmatrix}. \tag{14}$$

Note that the turbulent stresses introduce six additional unknowns into the averaged Navier-Stokes equation.

Phenomenological Theories of Turbulence

The classical phenomenological theories of turbulence are referred to as the first-order closures (closure at the order of the first moment) or zero-equation models (no additional differential equation are introduced to solve).

Boussineq Eddy Viscosity Model

Boussineq suggested

$$\tau_{ij}^{T} = -\frac{\rho \overline{u_k' u_k'}}{3}\delta_{ij} + \mu_T \left(\frac{\partial U_i}{\partial x_j} + \frac{\partial U_j}{\partial x_i} \right), \tag{15}$$

where

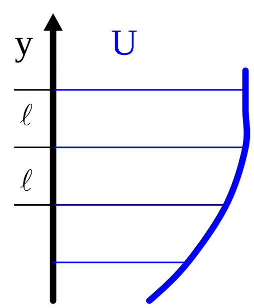
$$\mu_T = \rho \nu_T$$

is the eddy viscosity. For plane shear flow,

$$\tau_{12}^{T} = \tau^{T} = \rho \nu_T \frac{\partial U}{\partial y}. \tag{16}$$

It is now well recognized that ν_T is not a constant and is strong function of state of turbulent motion.

Prandtl Mixing Length Hypothesis



Prandtl argued that

$$\tau^{T} = -\rho \overline{u'v'} \tag{17}$$

and for a thin shear layer

$$(\overline{u'^2})^{\frac{1}{2}} \sim (\overline{v'^2})^{\frac{1}{2}} \sim \ell \frac{dU}{dy}, \tag{18}$$

where ℓ is the mixing length. Hence,

$$\tau^T = \rho\ell^2\left|\frac{\partial U}{\partial y}\right|\frac{\partial U}{dy}, \tag{19}$$

or

$$\upsilon_T = \ell^2\left|\frac{\partial U}{\partial y}\right|. \tag{20}$$

G.I. Taylor recovered the mixing length hypothesis based on a vorticity transport formulation and von Karman used a similarity analysis for estimating the mixing length as

$$\ell = \kappa\left|\frac{\frac{dU}{dy}}{\frac{d^2U}{dy^2}}\right| \tag{21}$$

The mixing length theory has dominated the field of turbulence modeling for more than half a century. It is now known that the mixing length hypothesis works for turbulent flows, which are characterized by single length and velocity scales. The reason for its effectiveness is simply due to dimensional requirements!

Logarithmic Velocity Near a Wall

Near a wall, there is a region (inertial sublayer) where turbulence is characterized by a single length scale (distance from the wall y) and a single velocity scale (shear velocity = $u^* = \sqrt{\frac{\tau_0}{\rho}}$). In this region,

$$\ell = \kappa y, \qquad \kappa = 0.4 = \text{von Karman constant} \tag{22}$$

and the shear stress is about τ_0. Equation (19), then implies

$$\tau_0 = \rho\kappa^2 y^2\left(\frac{\partial U}{\partial y}\right)^2, \tag{23}$$

or

$$\frac{dU}{dy} = \frac{u^*}{\kappa y}. \tag{24}$$

Integrating Equation (24), it follows that

$$\frac{U}{u^*} = U^+ = \frac{1}{\kappa}\ln y + c\,, \tag{25}$$

or

$$U^+ = \frac{1}{\kappa}\ln y^+ + B\,, \quad y^+ = \frac{u^* y}{\nu}\,, \qquad (30 < y^+ \le 300) \tag{26}$$

where $B \approx 5$.

Very near wall, in the viscous sublayer, turbulence fluctuation becomes small and the viscous stress becomes dominant. As a result,

$$\tau_0 = \mu\frac{dU}{dy} \tag{27}$$

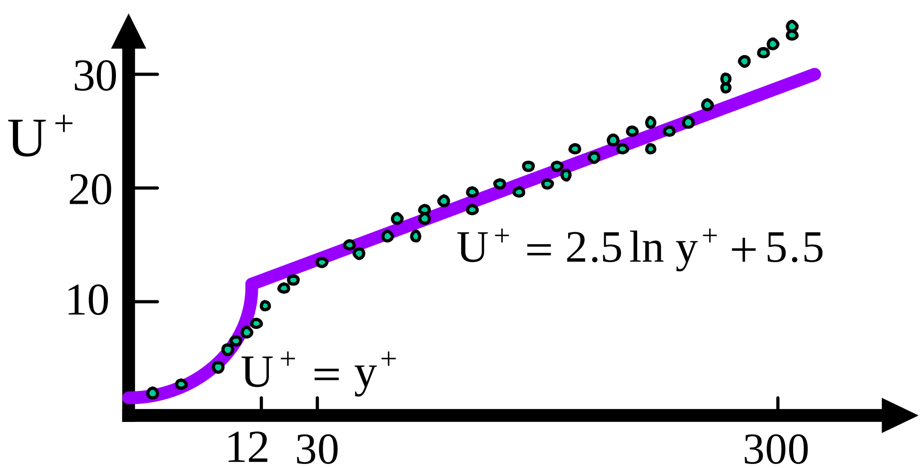
or

$$u^{*2} = \nu\frac{dU}{dy}\,, \qquad \frac{dU^+}{dy^+} = 1 \tag{28}$$

Hence,

$$U^+ = y^+ \qquad (0 < y^+ \le 5). \tag{29}$$

A schematics of the velocity profile is shown in the figure. Reichardt suggested a smooth curve for the buffer region (Schlichting, McGraw-Hill, 1960). Outside the viscous sublayer approximate expressions given by $u^+ = 8.74(y^+)^{\frac{1}{7}}$ and $u^+ = 11.5(y^+)^{\frac{1}{10}}$ may be used.



Schematics of turbulent velocity profile near a wall.