Creeping Flow Past a Sphere

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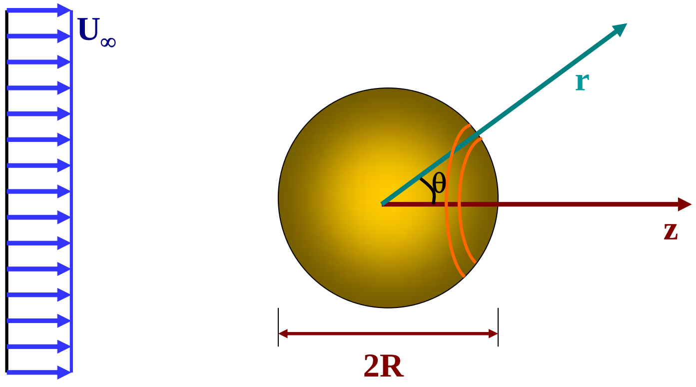
Outline

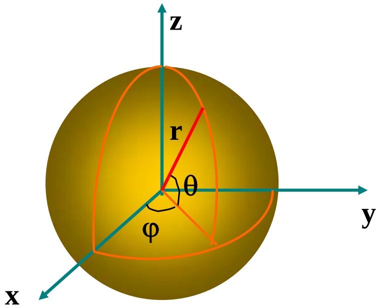
- Creeping Flow Equation
- Stream Function
- Boundary Conditions
- Pressure Variations
- Stokes Drag
- Oseen Drag
- Drag on a Droplet

Creeping Flow Past a Sphere



Spherical Coordinates

$$\begin{cases} x = r\cos\theta\cos\varphi \\ y = r\cos\theta\sin\varphi \\ z = r\sin\theta \end{cases}$$



Stream Function

$$v_r = \frac{1}{r^2\sin\theta}\frac{\partial\psi}{\partial\theta}$$

$$v_\theta = -\frac{1}{r\sin\theta}\frac{\partial\psi}{\partial r}$$

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Navier-Stokes Equation

$$\cancel{\frac{\partial}{\partial t}\left(E^2\psi\right)+\frac{1}{r^2\sin\theta}\frac{\partial\left(E^2\psi,\psi\right)}{\partial(r,\theta)}+\frac{2E^2\psi}{r^2\sin^2\theta}}\left(\frac{\partial\psi}{\partial r}\cos\theta-\frac{1}{r}\frac{\partial\psi}{\partial\theta}\sin\theta\right)=\nu E^4\psi$$

Creeping Flow →

$$E^4\psi=0$$

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Navier-Stokes Equation

$$\left[\frac{\partial^2}{\partial r^2}+\frac{\sin\theta}{r^2}\frac{\partial}{\partial\theta}\left(\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\right)\right]^2\psi=0$$

Boundary Conditions →

$$\left\{\begin{array}{ll} v_r=\frac{1}{r^2\sin\theta}\frac{\partial\psi}{\partial\theta}=0 & \text{at } r=R \\ v_\theta=-\frac{1}{r\sin\theta}\frac{\partial\psi}{\partial r}=0 & \text{at } r=R \\ \psi=\frac{1}{2}U_\infty r^2\sin^2\theta & \text{as } r\to\infty \end{array}\right\}$$

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Let →

$$\psi=f(r)\sin^2\theta$$

N-S →

$$\left(\frac{d^2}{dr^2}-\frac{2}{r^2}\right)\left(\frac{d^2}{dr^2}-\frac{2}{r^2}\right)f(r)=0$$

Solution →

$$f(r)=Ar^m$$

$$[(m-2)(m-3)-2][m(m-1)-2]=0$$

$$m=-1,1,2,4$$

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Solution →

$$f(r)=\frac{A}{r}+Br+Cr^2+Dr^4$$

Stream Function →

$$\psi=\left(\frac{1}{4}\frac{R^3}{r}-\frac{3}{4}Rr+\frac{1}{2}r^2\right)U_\infty\sin^2\theta$$

Velocity Field →

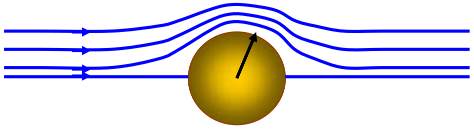
$$\frac{v_r}{U_\infty}=\left[1-\frac{3}{2}\frac{R}{r}+\frac{1}{2}\left(\frac{R}{r}\right)^3\right]\cos\theta$$

$$\frac{v_\theta}{U_\infty}=-\left[1-\frac{3}{4}\frac{R}{r}-\frac{1}{4}\left(\frac{R}{r}\right)^3\right]\sin\theta$$

Streamlines

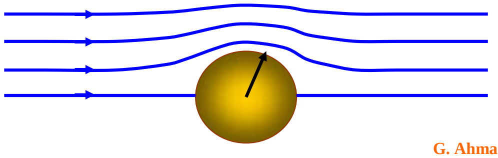
Viscous Flow

$$\psi = \left(\frac{1}{4}\frac{R^3}{r} - \frac{3}{4}Rr + \frac{1}{2}r^2\right)U_\infty \sin^2\theta$$



Potential Flow

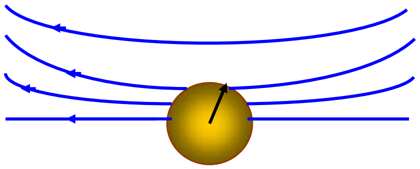
$$\psi = \frac{1}{2}U_\infty r^2 \sin^2\theta\left(1 - \frac{R^3}{r^3}\right)$$



Moving Sphere-Streamlines

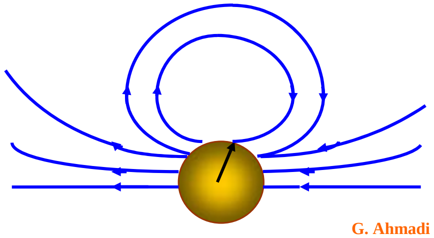
Viscous Flow

$$\psi = \left(\frac{1}{4}\frac{R^3}{r} - \frac{3}{4}Rr\right)U_\infty \sin^2\theta$$



Potential Flow

$$\psi = -\frac{1}{2}\frac{R^3}{r}U_\infty \sin^2\theta$$



Pressure and Drag

Navier-Stokes Equation

$$\frac{\partial P}{\partial r} = \frac{3\mu R U_\infty}{r^3}\cos\theta \qquad \frac{\partial P}{\partial \theta} = \frac{3\mu R U_\infty}{2r^2}\sin\theta$$

$$P = P_\infty - \frac{3\mu R U_\infty}{2r^2}\cos\theta$$

Shear Stress

$$\tau_{r\theta} = \mu\left(\frac{1}{r}\frac{\partial v_r}{\partial \theta} + \frac{\partial v_\theta}{\partial r}\right) = -\frac{U_\infty \mu \sin\theta}{r}\left(1 - \frac{3R}{4r} + \frac{5R^3}{4r^3}\right)$$

Stokes Drag

Drag

$$D = -\int_0^\pi (\tau_{r\theta}\,|_{r=R}\ \sin\theta + P\,|_{r=R}\ \cos\theta) 2\pi R^2 \sin\theta d\theta$$

$$D = 4\pi\mu U_\infty R + 2\pi\mu U_\infty R = 6\pi\mu U_\infty R$$

Drag Coefficient

$$C_D = \frac{D}{\frac{1}{2}\rho U_\infty^2 \pi R^2} = \frac{24}{Re} \qquad Re = \frac{\rho U_\infty (2R)}{\mu}$$

Oseen Drag

Oseen's Approximation

$$\mathbf{v}\cdot\nabla\mathrm{v} \approx U_\infty \frac{\partial \mathrm{v}}{\partial x}$$

N-S

$$\cancel{\frac{\partial \mathbf{v}}{\partial t}} + U_\infty \frac{\partial \mathbf{v}}{\partial x} = -\frac{1}{\rho}\nabla P + \nu\nabla^2\mathbf{v}$$

$$\nabla\cdot\mathbf{v} = 0$$

Drag Coefficient

$$C_D = \frac{24}{Re}\left[1 + \frac{3}{16}Re + \frac{9}{160}Re^2 \ln Re + \ldots\right]$$

Drag on a Cylinder

$$C_D = \frac{8\pi}{Re\left[0.5 - \gamma + \ln\left(\frac{8}{Re}\right)\right]}$$

$$\gamma = 0.577216\ldots$$

Empirical Formula

Drag on a Sphere

$$C_D = \frac{24}{Re}(1 + 0.15\,Re^{0.678}) \qquad 0 < Re \le 2\times10^5$$

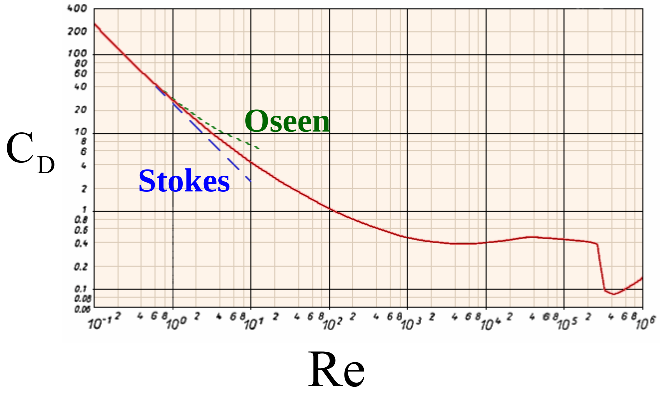
Drag on a Cylinder

$$C_D \approx 1 + 10\,Re^{-\frac{2}{3}} \qquad 1 < R < 2\times10^5$$

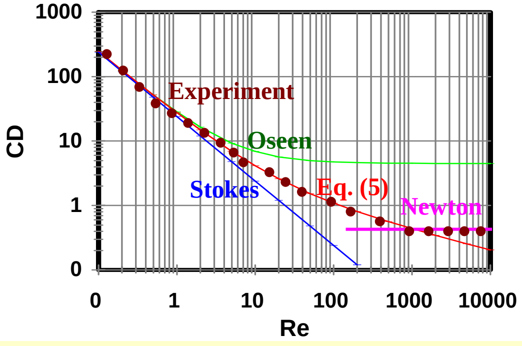
Drag on a Droplet

$$D = 6\pi\mu_0 U_\infty R \frac{1 + \frac{2\mu_0}{3\mu_d}}{1 + \frac{\mu_0}{\mu_d}}$$

Drag Force for a Sphere



Drag Force for a Sphere



Predictions of various models for drag coefficient for a spherical particle.

Concluding Remarks

- Creeping Flows
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- Pressure Variations
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Thank you!

Questions?