Clarkson University

HYDRODYNAMIC FORCES AND TORQUE FOR A NONSPHERICAL PARTICLE

In this section the hydrodynamic forces and torque for nonspherical particles are presented.

1. Drag on an Axisymmetric Body

Consider an axisymmetric body in creeping flow as shown in Figure 1. Here z and ρ are the axial and radial coordinate system and s and n are tangential and normal unit vectors.

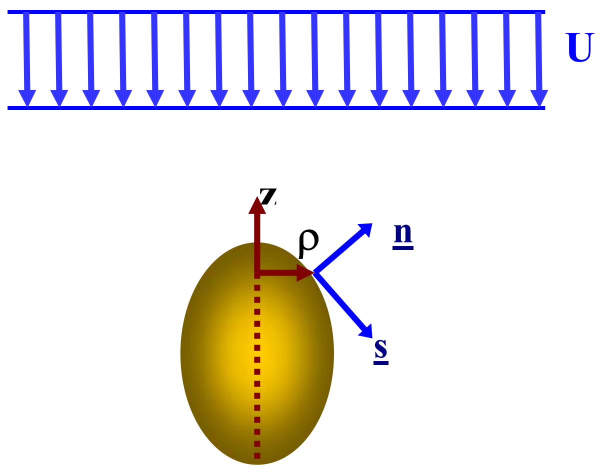

Figure 1. Schematics of an axisymmetric particle in a creeping flow.

The stress tensor for an incompressible viscous fluid is given as

$$\boldsymbol{\tau} = -p\mathbf{I} + 2\mu\mathbf{d}, \qquad \text{where} \qquad \mathbf{d} = \left[\nabla\mathbf{v} + (\nabla\mathbf{v})^T\right] \tag{1}$$

When an arbitrary shaped particle moves in a fluid, the resultant of the stresses acting on the surface of the body give rise to a hydrodynamic force $\mathbf{F}$ (drag, lift, and side force components and a hydrodynamic torque $\mathbf{T}$). In general these are given by

$$\mathbf{F} = \int_S \mathbf{dS}\cdot\boldsymbol{\tau}, \quad T = \int_S \mathbf{r}\times(\mathbf{dS}\cdot\boldsymbol{\tau}), \tag{2}$$

where S is the surface of the body and $\mathbf{r}$ is the position vector.

For an axisymmetric body in a uniform flow, the resultant of the stresses is only a drag force acting parallel to the direction of the flow. For the example shown in Figure 1,

$$F_z = \mathbf{F}\cdot\mathbf{e}_z = \int_S \mathbf{dS}\cdot\boldsymbol{\tau}\cdot\mathbf{e}_z = \int_S \mathbf{n}\cdot\boldsymbol{\tau}\cdot\mathbf{e}_z dS. \tag{3}$$

When an intrinsic coordinate is used,

$$dS = 2\pi\rho ds . \tag{4}$$

The expression for $\mathbf{n}\cdot\boldsymbol{\tau}$ as evaluated by Happel an Brenner is given by

$$\mathbf{n}\cdot\boldsymbol{\tau} = -\mathbf{n}p + 2\mu\left(\mathbf{n}\frac{\partial v_n}{\partial n} + \mathbf{s}\frac{\partial v_n}{\partial s}\right) + \mathbf{s}\mu\left(\frac{\partial v_s}{\partial n} - \frac{\partial v_n}{\partial s}\right) = -\mathbf{n}p - 2\mu\nabla\left(\frac{1}{\rho}\frac{\partial \psi}{\partial s}\right) + \mathbf{s}\frac{\mu}{\rho}E^2\psi \tag{5}$$

Using (5) in (3) after some algebra, it follows that

$$F_z = \mu\pi\int \rho^3 \frac{\partial}{\partial n}\left(\frac{E^2\psi}{\rho^2}\right)dS \tag{6}$$

This equation relates the Stokes stream function ψ to the drag for an axisymmetric body.

Evaluation of the integral on the right hand side of (6) could become quite cumbersome. When the fluid is at rest at infinity, the flow field for a point force is given by

$$\psi = \frac{F_z}{8\pi\mu}\frac{\rho^2}{r}, \quad p = -\frac{F_z}{4\pi}\frac{z}{r^3}, \tag{7}$$

where

$$r^2 = \rho^2 + z^2 . \tag{8}$$

Since far from the particle, the effect of the particle is essentially equivalent to a point, it follows that

$$F_z = 8\pi\mu \lim_{r\to\infty}\frac{r\psi}{\rho^2} . \tag{9}$$

When the fluid is not at rest at infinity, (9) is replaced by

$$F_Z = 8\pi\mu \lim_{r\to\infty}\frac{r(\psi - \psi_\infty)}{\rho^2} . \tag{10}$$

2. Oblate Spheroid in a Uniform Flow

Consider the creeping flow of a uniform stream passing an oblate spheroid at rest as shown in Figure 2.

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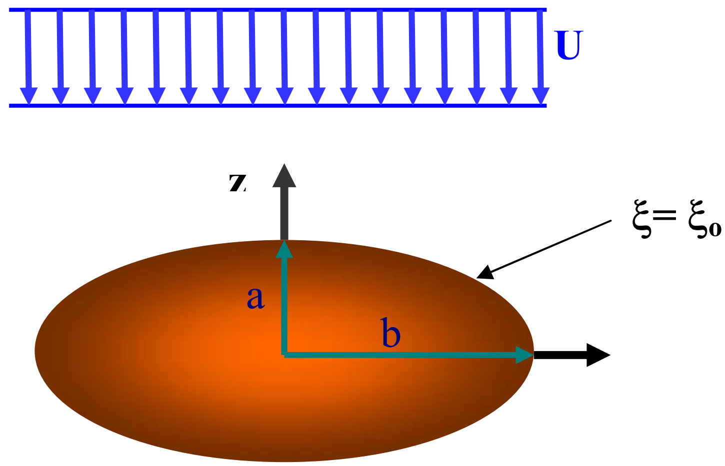

Figure 2. Schematic of an oblate spheroid in a uniform creeping flow.

The appropriate coordinate system is oblate spheroidal coordinates (ξ, θ, ϕ) with

$$\begin{Bmatrix} x = c\cosh\xi\sin\theta\cos\phi \\ y = c\cosh\xi\sin\theta\sin\phi \\ z = c\sinh\xi\cos\theta \end{Bmatrix} \tag{11}$$

For brevity, let

$$\lambda = \sinh\xi, \; \zeta = \cos\theta \tag{12}$$

Then ρ and z may be expressed as

$$\rho = \sqrt{x^2 + y^2} = c\cosh\xi\sin\theta = c\sqrt{\lambda^2 + 1}\sqrt{1 - \zeta^2}, \tag{13}$$

$$z = c\lambda\zeta. \tag{14}$$

The ranges of variation of λ and ζ then are

$$\infty > \lambda \geq 0 \qquad \text{and} \qquad 1 \geq \zeta \geq -1. \tag{15}$$

For obtaining the flow field around the spheroid, the biharmonic creeping flow equation given by

$$E^4\psi = 0 \tag{16}$$

must be solved. The boundary conditions are

$$\psi = 0 \qquad \text{at} \qquad \lambda = \lambda_0 \left(\xi = \xi_0\right)$$

$$\frac{\partial \psi}{\partial \lambda} = 0 \qquad \text{at} \qquad \lambda = \lambda_0 \left(\xi = \xi_0\right) \tag{17}$$

$$\psi \to \frac{1}{2}\rho^2 U = \frac{Uc^2}{2}\left(\lambda^2 + 1\right)\left(1 - \zeta^2\right) \ \text{ as } \ \lambda \ (\text{or } \xi) \to \infty . \tag{18}$$

Note that here

$$\frac{\partial}{\partial \xi} = \sqrt{\lambda^2 + 1}\,\frac{\partial}{\partial \lambda}, \qquad \frac{\partial}{\partial \theta} = -\sqrt{1 - \zeta^2}\,\frac{\partial}{\partial \zeta} \tag{19}$$

are used.

Using (19), the operator

$$E^2 = \frac{\partial^2}{\partial \rho^2} - \frac{1}{\rho}\frac{\partial}{\partial \rho} + \frac{\partial^2}{\partial z^2}, \tag{20}$$

may be restated as

$$E^2 = \frac{1}{c^2\left(\lambda^2 + \zeta^2\right)}\left[\left(\lambda^2 + 1\right)\frac{\partial^2}{\partial \lambda^2} + \left(1 - \zeta^2\right)\frac{\partial^2}{\partial \zeta^2}\right]. \tag{21}$$

The boundary conditions suggest a solution in the form,

$$\psi = \left(1 - \zeta^2\right) g(\lambda) \tag{22}$$

Substituting this solution into equation (16) and using (21), yields (Happel and Brenner)

$$\frac{\left(\lambda^2 + 1\right)\left(1 - \xi^2\right)}{C^4\left(\lambda^2 + \xi^2\right)}\left[4\left(G - \lambda G'\right) + \left(\lambda^2 + \rho^2\right)G''\right] = 0, \tag{23}$$

where

$$G(\lambda) = (\lambda^2 + 1)g''(\lambda) - 2g(\lambda) \tag{24}$$

Here

$$E^2\psi = \frac{1-\zeta^2}{C^2(\lambda^2 + \zeta^2)} G(\lambda) \tag{25}$$

was used. From (23) it follows that

$$4(G - \lambda G') + (\lambda^2 + \zeta^2)G'' = 0\,. \tag{26}$$

It is noted that the first term in (26) depends only on λ, while the second term depends on λ and ζ. The equation can be satisfied only if

$$G'' = 0 \tag{27}$$

and

$$G - \lambda G' = 0 \tag{28}$$

The solution satisfying both equations (27) and (28) is given as

$$G = C_1\lambda\,, \tag{29}$$

where C_1 is a constant. From (29) it follows that

$$(\lambda^2 + 1)g'' - 2g = C_1\lambda \tag{30}$$

The general solution to equation (30), which is the summation of the homogeneous and particular solutions, is given as

$$g(\lambda) = \underbrace{-\frac{1}{2}C_1\lambda}_{\text{particular solution}} + \underbrace{\frac{1}{2}C_2\left[\lambda - (\lambda^2 + 1)\cot^{-1}\lambda\right] + C_3(\lambda^2 + 1)}_{\text{Homogeneous solution}} \tag{31}$$

Using (31) in (22), the expression for the stream function becomes

$$\psi = (1 - \zeta^2)\left(-\frac{1}{2}C_1\lambda + \frac{1}{2}C_2\left[\lambda - (\lambda^2 + 1)\cot^{-1}\lambda\right] + C_3(\lambda^2 + 1)\right) \tag{32}$$

The constant C_1, C_2, and C_3 may now be determined by using the boundary conditions given by (17) and (18). These become

$$C_1 = \frac{2UC^2}{\lambda_0 - (\lambda_0^2 - 1)\cot^{-1}\lambda_0}$$

$$C_2 = -\frac{UC^2(\lambda_0^2 - 1)}{\lambda_0 - (\lambda_0^2 - 1)\cot^{-1}\lambda_0},$$

$$C_3 = \frac{1}{2}UC^2. \tag{33}$$

The final expression for the stream function becomes

$$\psi = \frac{1}{2}U\rho^2 \left\{ 1 - \frac{\frac{\lambda}{(\lambda^2+1)} - \left[\frac{(\lambda_0^2 - 1)}{(\lambda_0^2 + 1)}\right]\cot^{-1}\lambda}{\frac{\lambda_0}{(\lambda_0^2+1)} - \left[\frac{(\lambda_0^2 - 1)}{(\lambda_0^2 + 1)}\right]\cot^{-1}\lambda_0} \right\} \tag{34}$$

where $\lambda_0 = \sinh \xi_0$.

For an oblate spheroid translating with a velocity U in an otherwise quiscent fluid, the solution may be obtained by subtracting $\frac{1}{2}U\rho^2$ from equation (34), i.e.

$$\psi = -\frac{1}{2}U\rho^2 \frac{\frac{\lambda}{(\lambda^2+1)} - \left[\frac{(\lambda_0^2 - 1)}{(\lambda_0^2 + 1)}\right]\cot^{-1}\lambda}{\frac{\lambda_0}{(\lambda_0^2+1)} - \left[\frac{(\lambda_0^2 - 1)}{(\lambda_0^2 + 1)}\right]\cot^{-1}\lambda_0}. \tag{35}$$

The force exerted on the spheroids can be obtained by using (34) or (35) in Equations (8) or (9). Since $r \to c\lambda$ at large distances from the spheroid

$$F_z = 8\pi\mu c \lim_{\lambda\to\infty} \frac{\lambda\psi}{\rho^2}. \tag{36}$$

It then follows that

$$F_z = -\frac{8\pi\mu c U}{\lambda_0 - (\lambda_0^2 - 1)\cot^{-1}\lambda_0}. \tag{37}$$

Equation (37) may be expressed as

$$F_Z = -6\pi\mu a U K \tag{38}$$

where

$$K = \frac{1}{\left\{\frac{3}{4}\sqrt{\lambda_0^2+1}\left[\lambda_0 - \left(\lambda_0^2-1\right)\cot^{-1}\lambda_0\right]\right\}} \tag{39}$$

is the shape factor. Note that $c=\sqrt{a^2-b^2}$ and $\lambda_o = \frac{b}{c} = \frac{1}{\sqrt{(a/b)^2-1}}$.

Circular Disk

The solution for a disk of radius a moving perpendicular to its plane as shown in Figure 3 is obtained by letting $\lambda_0 \to 0$ in equation (35). i.e.,

$$\psi = -\frac{U\rho^3}{\pi}\left(\frac{\lambda}{\lambda^2+1} + \cot^{-1}\lambda\right). \tag{40}$$

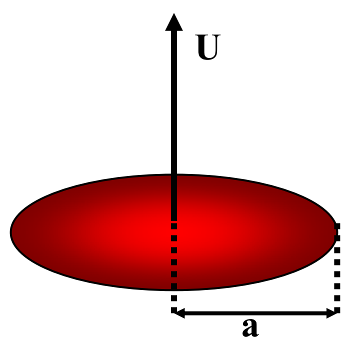

Figure 3. Schematic of a circular disk in creeping flow motion.

The corresponding force acting on the disk is given as

$$F_z = -16\mu a U \tag{41}$$

(For more details see Happel & Brenner "Low Reynolds Number Hydrodynamics," pp. 145-149.)

3. Prolate Spheroid Translating in a Quiescent Fluid

The motion of a rigid prolate spheroid parallel to its axis of revolution as shown in Figure 4 is studied in this section. The appropriate coordinates system for this problem is the prolate coordinate system (ξ, θ, ϕ) with

$$\begin{Bmatrix} x = c\sinh\xi\sin\theta\cos\psi \\ y = c\sinh\xi\sin\theta\sin\psi \\ z = c\cosh\xi\cos\theta \end{Bmatrix}. \tag{42}$$

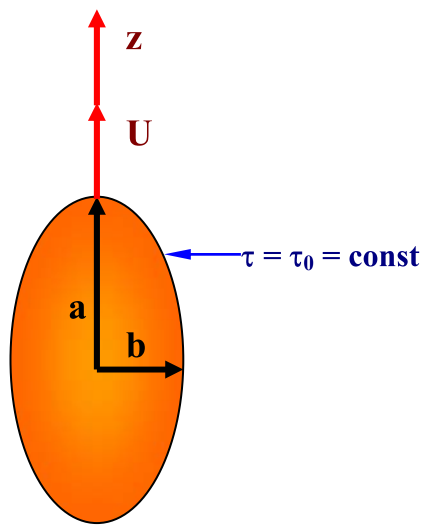

Figure 4. Schematic of a prolate spheroids in creeping flow motion.

For convenience, we let

$$\tau = \cosh\xi, \quad \zeta = \cos\theta. \tag{43}$$

The surface $\tau = \text{const}$. ($\xi = \text{const}$) are prolate spheroids. Then ρ and z may be expressed as

$$\rho = \sqrt{\tau^2 - 1}\sqrt{1 - \zeta^2}\, c \tag{44}$$

and

$$z = c\tau\zeta \tag{45}$$

Similar to the method used for the oblate spheroid, one can solve the equation of creeping motion in prolate spheroidal coordinates subject to appropriate boundary conditions. The stream function for a prolate spheroid translating with velocity U in the positive z direction parallel to its axis of revolution is given as

$$\psi = -\frac{1}{2}U\rho^2 \frac{\left[\frac{(\tau_0^2+1)}{(\tau_0^2-1)}\right]\coth^{-1}\tau - \left[\frac{\tau}{(\tau^2-1)}\right]}{\left[\frac{(\tau_0^2+1)}{(\tau_0^2-1)}\right]\coth^{-1}\tau_0 - \left[\frac{\tau}{(\tau_0^2-1)}\right]}. \tag{46}$$

Using this expression in Equation (8), one obtains the force acting on the prolate spheroid, that is

$$F_z = -\frac{8\pi\mu c U}{(\tau_0^2+1)\coth^{-1}\tau_0 - \tau_0} \tag{47}$$

where

$$c = \sqrt{a^2 - b^2} \tag{48}$$

and $\tau_o = \cosh\xi_o = \frac{a}{c} = \frac{1}{\sqrt{1-(b/a)^2}}$. Equation (47) may be restated as

$$F_z = -6\pi\mu b U K \tag{49}$$

where the shape factor k is given by

$$K = \left\{\frac{3}{4}\sqrt{\tau_0^2 - 1}\left[(\tau_0^2+1)\coth^{-1}\tau_0 - \tau_0\right]\right\}^{-1} \tag{50}$$

Elongated Rod

When the major axis, a, is much greater than its equatorial radius, b, the spheroid resembles a long thin rod. For this limiting case

$$F_Z = -\frac{4\pi\mu_a U}{\ln\left(\frac{a}{b}\right) + \ln 2 - \frac{1}{2}}.$$

(For more detail see Happel & Brenner "Low Reynolds Number Hydrodynamics" pp. 154-156.)