Exact Solutions to the Navier-Stokes Equation

Unsteady Parallel Flows (Plate Suddenly Set in Motion)

Consider that special case of a viscous fluid near a wall that is set suddenly in motion as shown in Figure 1. The unsteady Navier-Stokes reduces to

$$\frac{\partial u}{\partial t} = \nu \frac{\partial^2 u}{\partial y^2} \tag{1}$$

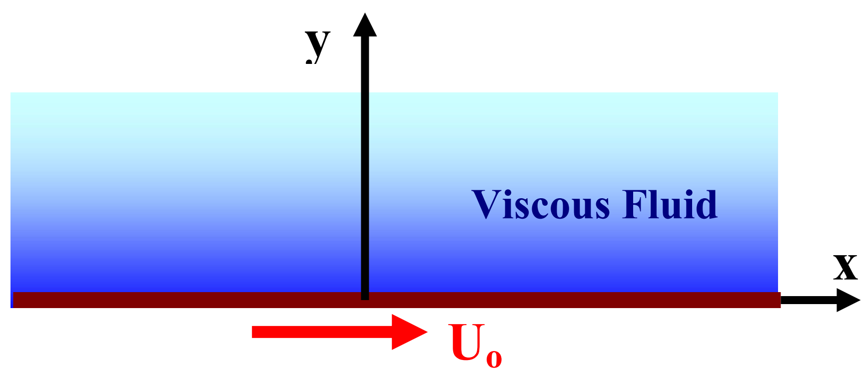

Figure 1. Schematics of flow near a wall suddenly set in motion.

The boundary conditions are:

$$\text{At } y = 0 \qquad u = U_0 \tag{2}$$

$$\text{at } y = \infty, \qquad u = 0 \tag{3}$$

The corresponding initial condition for the fluid that starts from rest is given as

$$\text{at } t = 0 \qquad u = 0. \tag{4}$$

Similarity Solution (Group Theory)

Let

$$t \sim t^1, \qquad y \sim t^a, \tag{5}$$

Equation (1) implies that

$$1 = 2a, \to \qquad a = \frac{1}{2}, \tag{6}$$

Thus,

$$y \sim t^{\frac{1}{2}} \tag{7}$$

Now introducing the similarity variables

$$\eta = \frac{y}{2\sqrt{\nu t}}, \qquad \frac{u}{U_0} = f(\eta), \tag{8}$$

we find

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial \eta}\frac{1}{2\sqrt{\nu t}}, \qquad \frac{\partial^2 u}{\partial y^2} = \frac{\partial^2 u}{\partial \eta^2}\frac{1}{4\nu t} \tag{9}$$

$$\frac{\partial u}{\partial t} = \frac{\partial u}{\partial \eta}\frac{y}{2\sqrt{\nu t}}\left(-\frac{1}{2t}\right) = -\frac{\partial u}{\partial \eta}\frac{\eta}{2t}. \tag{10}$$

Substituting (9) and (10) in Equation (1), we find

$$-f'\frac{\eta}{2t} = \nu f''\frac{1}{4\nu t}, \tag{11}$$

or

$$f'' + 2\eta f' = 0 \tag{12}$$

Boundary and initial conditions (2)-(4) in terms of the similarity variables become

$$f(0) = 1, \qquad f(\infty) = 0. \tag{13}$$

From Equation (12), it follows that

$$\frac{f''}{f'} = -2\eta, \qquad \text{or} \qquad \ln f' = \ln c - \eta^2 \tag{14}$$

or

$$f' = ce^{-\eta^2}, \qquad \text{and} \qquad f = c\int_0^{\eta} e^{-\eta_1^2}\, d\eta_1 + 1, \tag{15}$$

where the first boundary condition in (13) is used. The second boundary condition implies that

$$f(\infty) = 0 = 1 + c\int_0^\infty e^{-\eta_1^2} d\eta_1 \qquad \text{or} \quad c = -\frac{1}{\int_0^\infty e^{-\eta_1^2} d\eta_1} = -\frac{2}{\sqrt{\pi}} \tag{16}$$

Equation (15) then becomes

$$f = 1 - \frac{2}{\sqrt{\pi}}\int_0^\eta e^{-\eta_1^2} d\eta_1 = 1 - \operatorname{erf}(\eta) \tag{17}$$

or

$$f = \operatorname{erfc}\eta, \qquad u = U_0 \operatorname{erfc}\left(\frac{y}{2\sqrt{\nu t}}\right) \tag{18}$$

Time variations of the velocity profile as predicted by Equation (18) are shown in Figure 2.

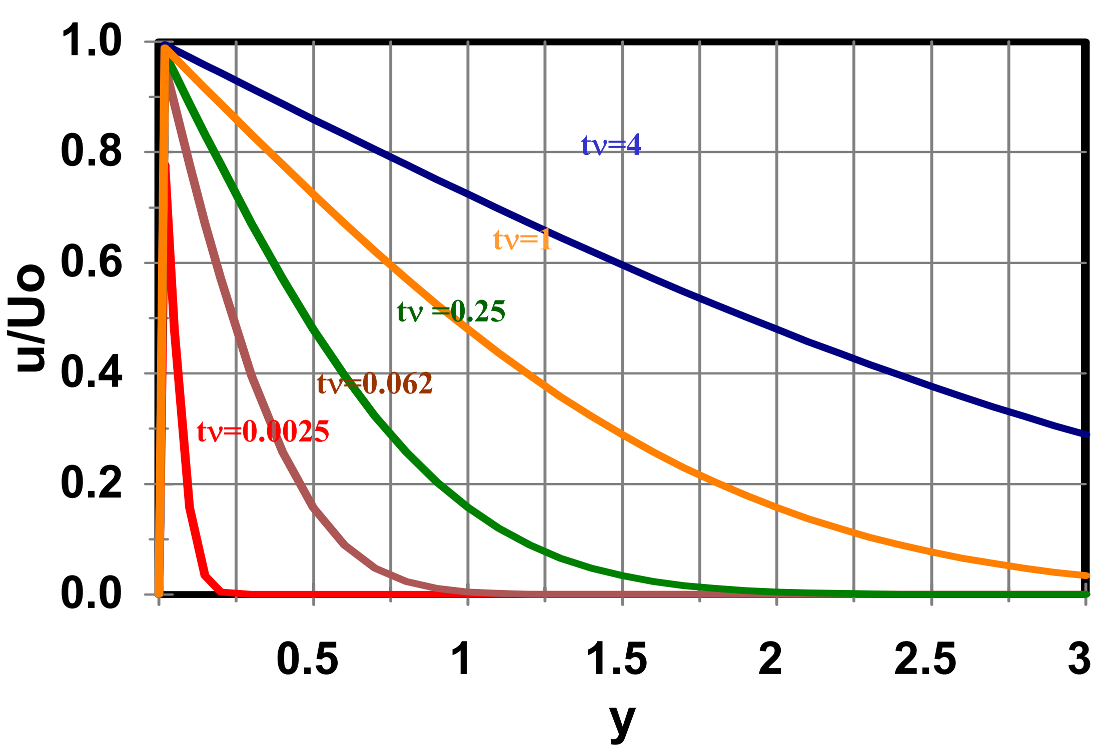

Figure 2. Time variations of velocity profile.

An alternative is to use the transform method. Taking Laplace transform of Equation (1), it follows that

$$s\bar{u} = \nu \frac{\partial^2 \bar{u}}{\partial y^2} \tag{19}$$

or

$$\bar{u}'' - \frac{s}{\nu}\bar{u} = 0 \tag{20}$$

The solution to (20) is

$$\bar{u} = Ae^{-\sqrt{\frac{s}{\nu}}y} + Be^{\sqrt{\frac{s}{\nu}}y} \tag{21}$$

Boundary conditions (2) and (3) imply that

$$A = \frac{U_0}{s}, \qquad B = 0 \tag{22}$$

Thus, the solution in the transform domain is given by

$$\bar{u} = \frac{U_0}{s} e^{-\sqrt{\frac{s}{\nu}}y} \tag{23}$$

Inverse Laplace transform of (23) gives

$$u = U_0 \operatorname{erfc}\left(\frac{y}{2\sqrt{\nu t}}\right). \tag{24}$$

Oscillating Plate

Consider that case of a viscous fluid near an oscillating wall as shown in Figure 3. The unsteady Navier-Stokes reduces to

$$\frac{\partial u}{\partial t} = \nu \frac{\partial^2 u}{\partial y^2} \tag{25}$$

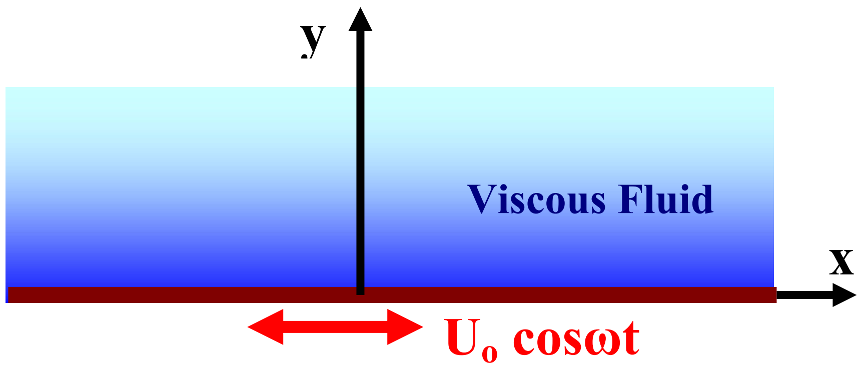

Figure 2. Schematics of flow near an oscillating wall.

The boundary conditions are:

$$u = U_0 \cos \omega t \qquad \text{at } y = 0 \tag{26}$$

$$u = 0 \qquad \text{at } y = \infty \tag{27}$$

Let

$$u = U_0 e^{-ky} \cos(\omega t - ay). \tag{28}$$

Then

$$\frac{\partial u}{\partial t} = -\omega U_0 e^{-ky} \sin(\omega t - ay) \tag{29}$$

$$\frac{\partial u}{\partial y} = U_0 e^{-ky} \left(-k \cos(\omega t - ay) + a \sin(\omega t - ay)\right) \tag{30}$$

$$\frac{\partial^2 u}{\partial y^2} = U_0 e^{-ky} \left(k^2 \cos\theta - 2ka \sin\theta - a^2 \cos\theta\right), \;\; \theta = \omega t - ay \tag{31}$$

Substituting (29)-(31) into Equation (25) it follows that

$$-\omega \sin\theta = \nu\left(\left(k^2 - a^2\right)\cos\theta - 2ak \sin\theta\right) \tag{32}$$

or

$$a^2 = k^2 \tag{33}$$

$$\omega = 2ak\nu = 2k^2\nu \tag{34}$$

$$k = \sqrt{\frac{\omega}{2\nu}} = a \tag{35}$$

Thus, the velocity profile is given as

$$u = U_0 e^{-ky}\cos(\omega t - ky), \qquad k = \sqrt{\frac{\omega}{2\nu}}. \tag{36}$$

Unsteady Flow in a Tube

Consider flow in a circular tube subject to a step change in pressure as shown in Figure 4. The Navier-Stokes equation reduces to

$$\frac{\partial v_z}{\partial t} = -\frac{1}{\rho}\frac{dP}{dz} + \nu\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_z}{\partial r}\right) \tag{37}$$

Introducing dimensionless variables,

$$\xi = \frac{r}{R}, \quad \tau = \frac{\mu t}{\rho R^2} = \frac{\nu t}{R^2}, \quad v_z = -\frac{1}{4\mu}\frac{dP}{dz}R^2\varphi(\xi), \tag{38}$$

we find

$$\frac{\partial \varphi}{\partial \tau} = 4 + \frac{1}{\xi}\frac{\partial}{\partial \xi}\left(\xi\frac{\partial \varphi}{\partial \xi}\right). \tag{39}$$

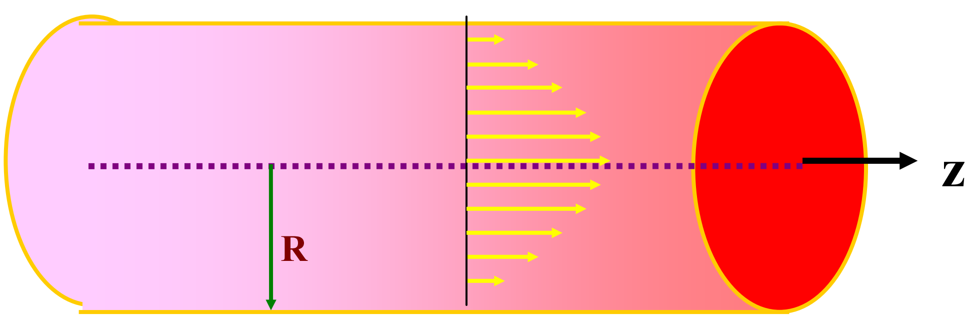

Figure 4. Schematics of flow in a tube subject to step change in pressure.

The boundary condition is

$$\varphi = 0 \text{ at } \xi = 1, \tag{40}$$

with the initial conditions

$$\varphi = 0 \text{ at } \tau = 0. \tag{41}$$

Let

$$\phi = 1 - \xi^2 - \psi, \tag{42}$$

Equation (39) reduces to

$$\frac{\partial \psi}{\partial \tau} = \frac{1}{\xi}\frac{\partial}{\partial \xi}\left(\xi \frac{\partial \psi}{\partial \xi}\right) \tag{43}$$

The boundary and initial conditions (40) and (41) now become

$$\text{At } \xi = 1, \qquad \psi = 0. \tag{44}$$

$$\text{At } \tau = 0, \qquad \psi = 1 - \xi^2. \tag{45}$$

To find the solution the method of separation of variable is used. That is let

$$\psi = F(\xi)T(\tau) \tag{46}$$

Equation (43) then becomes

$$\frac{\dot{T}}{T} = \frac{1}{F\xi}\frac{d}{d\xi}\left(\xi \frac{dF}{d\xi}\right) = -\alpha^2. \tag{47}$$

From Equation (47), it follows that

$$\dot{T} + \alpha^2 T = 0, \tag{48}$$

$$\xi^2 \frac{d^2F}{d\xi^2} + \xi \frac{dF}{d\xi} + \alpha^2\xi^2 F = 0\,. \tag{49}$$

The solutions to Equations (48) and (49) are given as

$$T = Ce^{-\alpha^2\tau} \tag{50}$$

$$F = AJ_0(\alpha\xi) + BY_0(\alpha\xi)\,, \tag{51}$$

where $J_0(\alpha\xi)$ and $Y_0(\alpha\xi)$ are Bessel function of first and second kind of zeroth order. The boundary conditions are

$$F(0) \sim \text{finite} \Rightarrow B = 0 \quad \text{since} \quad Y_0(0) \to \infty\,. \tag{52}$$

and

$$F(1) = 0 \Rightarrow J_0(\alpha) = 0\,. \tag{53}$$

Equation (53) is a characteristic equation. The corresponding eigenvalues, α_n, are given as

$$\alpha_1 = 2.405\,,\ \alpha_2 = 5.52\,,\ \alpha_3 = 8.654\,,\ldots \tag{54}$$

The general solution for Equation (43) then is given by

$$\psi = \sum_n A_n e^{-\alpha_n^2\tau} J_0(\alpha_n\xi) \tag{55}$$

Using the initial condition

$$1 - \xi^2 = \sum_n A_n J_0(\alpha_n\xi) \tag{55}$$

then

$$A_n = \frac{\int_0^1 (1-\xi^2)\xi J_0(\alpha_n\xi)d\xi}{\int_0^1 \xi J_0^2(\alpha_n\xi)d\xi} = \frac{4J_1(\alpha_n)/\alpha_n^3}{0.5J_1^2(\alpha_n)} \tag{56}$$

or

$$A_n = \frac{8}{\alpha_n^3 J_1(\alpha_n)}\,. \tag{57}$$

Hence,

$$\psi = 8\sum_n \frac{e^{-\alpha_n^2\tau} J_0(\alpha_n \xi)}{\alpha_n^3 J_1(\alpha_n)}, \tag{58}$$

and

$$\varphi = 1 - \xi^2 - 8\sum_n \frac{J_0(\alpha_n \xi)}{\alpha_n^3 J_1(\alpha_n)} e^{-\alpha_n \tau} \tag{59}$$

Variation of the velocity profile in the pipe is shown schematically in Figure 5.

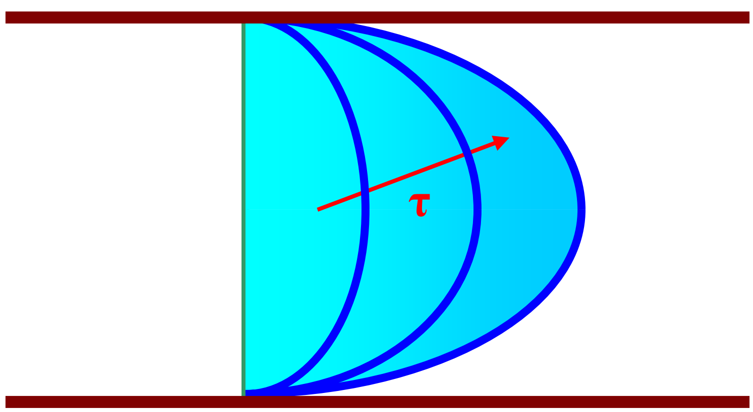

Figure 5. Variations of velocity field in a tube subject to a step change in pressure.

Noncircular Pipe Flows

Consider steady state viscous flows in a pipe with arbitrary cross section under a constant pressure gradient as shown in Figure 6. The Navier-Stokes equation is given as

$$\nabla^2 W = \frac{1}{\mu}\frac{dP}{dz} = \text{const}. \tag{60}$$

The corresponding boundary condition is

$$W = 0 \qquad \text{on} \qquad S. \tag{61}$$

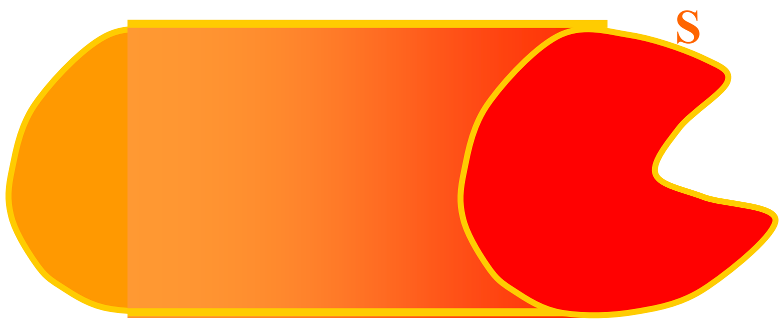

Figure 6. An arbitrary cross-section pipe subject to a constant pressure gradient.

Elliptical Pipes

Consider an elliptical cross-section pipe shown in Figure 7 with its boundary given as

$$\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1. \tag{62}$$

We assume that the velocity field is given by

$$w = A\left(1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}\right). \tag{63}$$

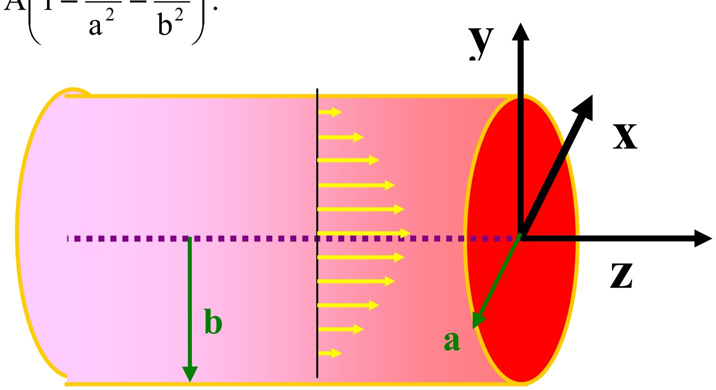

Figure 7. Elliptical cross-section pipe subject to a constant pressure gradient.

$$\nabla^2 w = -A\left(\frac{2}{a^2}+\frac{2}{b^2}\right) = -\frac{2A\left(a^2+b^2\right)}{a^2 b^2} = \frac{1}{\mu}\frac{dP}{dz} \tag{64}$$

Hence

$$A = -\frac{1}{2\mu}\frac{dP}{dz}\frac{a^2 b^2}{a^2+b^2}\left(1-\frac{x^2}{a^2}-\frac{y^2}{b^2}\right) \tag{65}$$

The flow rate is given as

$$Q = \iint w\,dx\,dy\,. \tag{66}$$

After integration, it follows that

$$Q = -\frac{\pi}{4\mu}\frac{dP}{dz}\frac{a^3 b^3}{a^2+b^2}\,. \tag{67}$$

Triangular Pipes

Consider a pipe as shown in Figure 8 whose cross section is an equilateral triangle. The equation of the section is given as

$$f(x,y) = (x-a)\left(x-\sqrt{3}y+2a\right)\left(x+\sqrt{3}y+2a\right) = 0\,. \tag{68}$$

Assuming

$$w = Af(x,y) \tag{69}$$

Then

$$\nabla^2 w = A\nabla^2 f(x,y) = 12aA = \frac{1}{\mu}\frac{dP}{dz}\,. \tag{70}$$

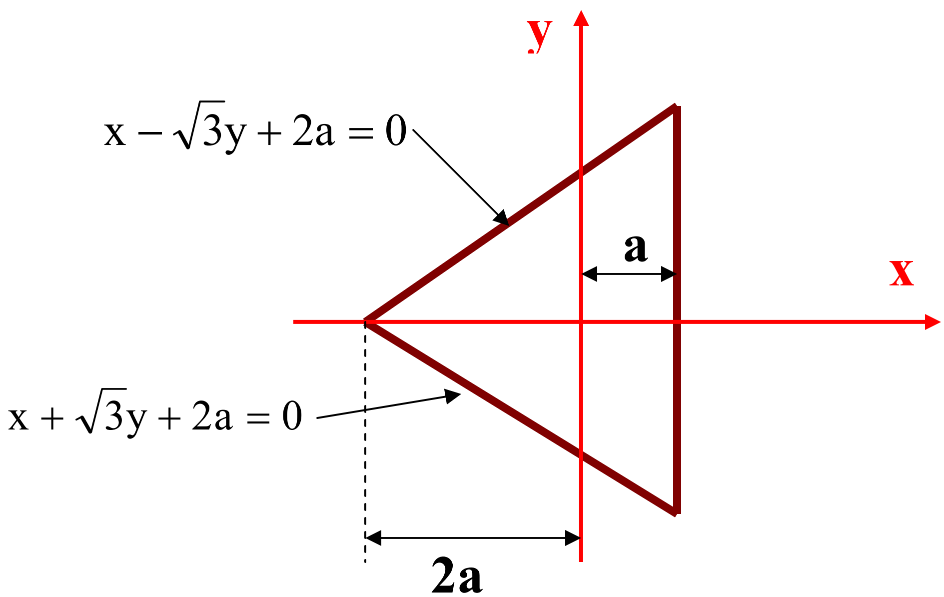

Figure 8. A triangular pipe subject to a constant pressure gradient.

Thus,

$$A = \frac{1}{12\mu a}\frac{dP}{dx} \tag{71}$$

Hence,

$$w = \frac{1}{12\mu a}\frac{dP}{dx}(x-a)\left(x-\sqrt{3}y+2a\right)\left(x+\sqrt{3}y+2a\right) \tag{72}$$