Creeping Flow Past a Sphere

Consider a creeping flow past a sphere as shown in the figure.

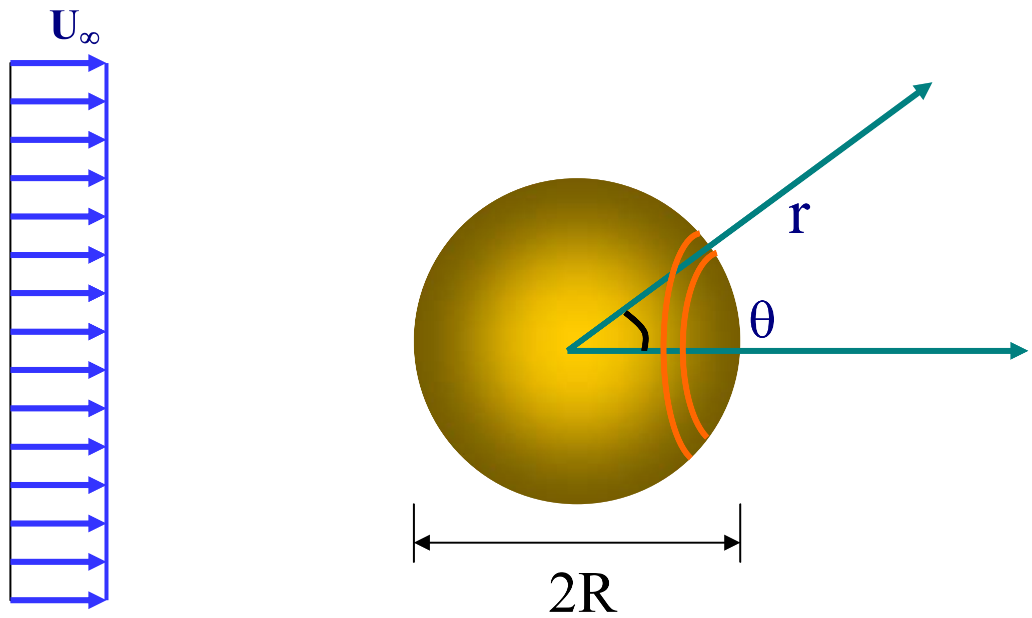


Figure 1. Schematics of creeping flow over a sphere.

The equations governing a fluid creeping motion is given as

$$E^4\psi = 0\,, \tag{1}$$

or

$$\left[\frac{\partial^2}{\partial r^2} + \frac{\sin\theta}{r^2}\frac{\partial}{\partial\theta}\left(\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\right)\right]^2 \psi = 0 \tag{2}$$

For flow around the sphere shown in the figure, the boundary conditions are:

$$\left\{\begin{array}{ll} v_r = \dfrac{1}{r^2\sin\theta}\dfrac{\partial\psi}{\partial\theta} = 0 & \text{at } r = R \\ v_\theta = -\dfrac{1}{r\sin\theta}\dfrac{\partial\psi}{\partial r} = 0 & \text{at } r = R \\ \psi = \dfrac{1}{2}U_\infty r^2 \sin^2\theta & \text{as } r \to \infty \end{array}\right\} \tag{3}$$

Now let

$$\psi = f(r)\sin^2\theta \tag{4}$$

Equation (2) then becomes

$$\left(\frac{d^2}{dr^2} - \frac{2}{r^2}\right)\left(\frac{d^2}{dr^2} - \frac{2}{r^2}\right) f(r) = 0 \tag{5}$$

Equation (4) is an Euler differential equation, which accepts a power law solution. That is

$$f(r) = Ar^m \tag{6}$$

Substituting (6) into (5), it follows that

$$[(m-2)(m-3)-2][m(m-1)-2] = 0 \tag{7}$$

Roots of the characteristic equation given by (7) are

$$m = -1, 1, 2, 4 \tag{8}$$

Thus, the general solution is given as

$$f(r) = \frac{A}{r} + Br + Cr^2 + Dr^4 \tag{9}$$

Using (4), the boundary conditions given by Equation (3) becomes

$$f(R) = f'(R) = 0 \tag{10}$$

$$f(\infty) \to \frac{1}{2} U_\infty r^2 \tag{11}$$

Comparing the expression given by (9) as $r \to \infty$ with (12) implies that

$$D = 0\,,\ C = \frac{1}{2} U_\infty \tag{12}$$

Using boundary conditions given by (10), it follows that

$$A = \frac{1}{4}U_\infty R^3,\ B = -\frac{3}{4}U_\infty R \tag{13}$$

Substituting (12) and (13) into (9), the explicit expression for the stream function becomes

$$\psi = \left(\frac{1}{4}\frac{R^3}{r} - \frac{3}{4}Rr + \frac{1}{2}r^2\right)U_\infty \sin^2\theta \tag{14}$$

The velocity components are then given by

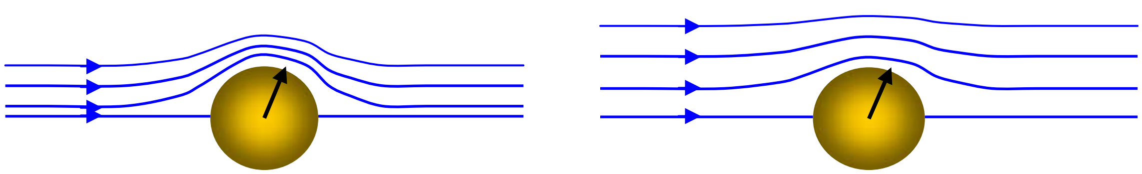
$$\frac{v_r}{U_\infty} = \left[1 - \frac{3}{2}\frac{R}{r} + \frac{1}{2}\left(\frac{R}{r}\right)^3\right]\cos\theta \tag{15}$$

$$\frac{v_\theta}{U_\infty} = -\left[1 - \frac{3}{4}\frac{R}{r} - \frac{1}{4}\left(\frac{R}{r}\right)^3\right]\sin\theta \tag{16}$$

Figure 2a shows the streamline for the creeping flow around a sphere. Comparing the streamlines of the creeping flow conditions to the potential flow one given by

$$\psi = \frac{1}{2}U_\infty r^2 \sin^2\theta\left(1 - \frac{R^3}{r^3}\right) \tag{17}$$

and is plotted in Figure 2 b, it appears that the stream lines are more dispersed.

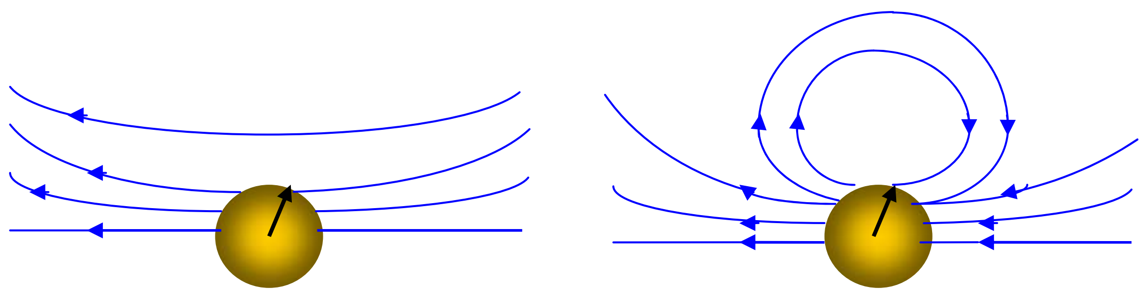


a) Viscous Flow   b) Potential Flow

Figure 2. Comparison of the streamlines for creeping and potential flows.

For moving spheres, the stream function is given by

$$\psi|_{moving} = \psi - \frac{1}{2}U_\infty r^2 \sin^2\theta \tag{18}$$



a) Viscous Flow

b) Potential Flow

Figure 3. Comparison of the streamlines for creeping potential flows in a moving frame..

For the moving sphere coordinates the corresponding streamlines are shown in Figure 3. Figure 3a shows that the particle appears to be dragging the viscous fluid as it moves, while Figure 3b through suggests that the particle pushes the fluid in the potential flow regime.

Pressure, Shear Stress and Drag Variations

Variation of the pressure filed is studied in this section. The Navier-Stokes equation under creeping motion assumption in spherical coordinate system is given as

$$\frac{1}{\mu}\frac{\partial P}{\partial r}=\left(\frac{\partial^2 v_r}{\partial r^2}+\frac{2}{r}\frac{\partial v_r}{\partial r}+\frac{1}{r^2}\frac{\partial^2 v_r}{\partial \theta^2}+\frac{\cot\theta}{r^2}\frac{\partial v_r}{\partial \theta}-\frac{2v_r}{r^2}\right)-\frac{2}{r^2}\left(\frac{\partial v_\theta}{\partial \theta}+\cot\theta\, v_\theta\right) \tag{19}$$

$$\frac{1}{\mu r}\frac{\partial P}{\partial \theta}=\frac{\partial^2 v_\theta}{\partial r^2}+\frac{2}{r}\frac{\partial v_\theta}{\partial r}+\frac{1}{r^2}\frac{\partial^2 v_\theta}{\partial \theta^2}+\frac{\cot\theta}{r^2}\frac{\partial v_\theta}{\partial \theta}-\frac{v_\theta}{r^2\sin^2\theta}+\frac{2}{r^2}\frac{\partial v_r}{\partial \theta} \tag{20}$$

Using the expression for the velocity components as given by Equations (15) and (16), after some algebra we find

$$\frac{\partial P}{\partial r}=\frac{3\mu R U_\infty}{r^3}\cos\theta \tag{21}$$

$$\frac{\partial P}{\partial \theta}=\frac{3\mu R U_\infty}{2r^2}\sin\theta \tag{22}$$

Integrating (21) and (22), it follows that

$$P = P_\infty - \frac{3\mu R U_\infty}{2r^2}\cos\theta \tag{23}$$

The expression for shear stress is given by

$$\tau_{r\theta} = \mu\left(\frac{1}{r}\frac{\partial v_r}{\partial\theta} + \frac{\partial v_\theta}{\partial r}\right) = -\frac{U_\infty \mu \sin\theta}{r}\left(1 - \frac{3R}{4r} + \frac{5R^3}{4r^3}\right) \tag{24}$$

The drag force acting on the sphere then is given by

$$D = -\int_0^\pi \left(\tau_{r\theta}\,|_{r=R}\ \sin\theta + P\,|_{r=R}\ \cos\theta\right) 2\pi R^2 \sin\theta d\theta \tag{25}$$

After integration, the famous Stokes drag law follows. That is

$$D = 4\pi\mu U_\infty R + 2\pi\mu U_\infty R = 6\pi\mu U_\infty R \tag{26}$$

The corresponding drag coefficient is given as

$$C_D = \frac{D}{\frac{1}{2}\rho U_\infty^2 \pi R^2} = \frac{24}{Re} \tag{27}$$

where the Reynolds number is defined as

$$Re = \frac{\rho U_\infty (2R)}{\mu} \tag{28}$$

Oseen's Improvement

Oseen's creeping flow equation includes an approximate inertia term. That is he simplified the Navier-Stokes equation by assuming a constant axial velocity in the inertia term. i.e.,

$$\mathbf{v}\cdot\nabla v \approx U_\infty \frac{\partial v}{\partial x} \tag{29}$$

The so-called Oseen equation then is given as

$$\frac{\partial \mathbf{v}}{\partial t} + U_\infty \frac{\partial \mathbf{v}}{\partial x} = -\frac{1}{\rho}\nabla P + \nu\nabla^2 \mathbf{v} \tag{30}$$

Oseen then solved Equation (30) together with the continuity equation given by

$$\nabla \cdot \mathbf{v} = 0 \tag{31}$$

He found the following correction for the drag coefficient:

$$C_D = \frac{24}{Re}\left[1 + \frac{3}{16}Re + \frac{9}{160}Re^2 \ln Re + \ldots\right] \tag{32}$$

Oseen Drag for a Cylinder

The creeping flow equation does not accept a solution for flow over a cylinder. The Oseen approximation, however, converges and leads to the following expression for the drag coefficient for a cylinder:

$$C_D = \frac{8\pi}{Re\left[0.5 - \gamma + \ln\left(\frac{8}{Re}\right)\right]} \qquad \text{(cylinder)} \qquad Re \le 1 \tag{33}$$

Here $\gamma = 0.577216...$ is the Euler number.

Empirical Formula

For larger Reynolds number, the following empirical equations for spherical and cylindrical particles may be used:

$$C_D = \frac{24}{Re} + \frac{6}{1+\sqrt{Re}} + 0.4 \qquad \text{(sphere)} \qquad 0 < Re \le 2\times 10^5 \tag{34}$$

$$C_D = \frac{24}{Re}(1 + 0.15\,Re^{0.678}) \qquad \text{(sphere)} \qquad 0 < Re \le 2\times 10^5 \tag{35}$$

$$C_D \approx 1 + 10\,Re^{-\frac{2}{3}} \qquad \text{(cylinder)} \qquad 1 < R < 2\times 10^5 \tag{36}$$

Drag on a Droplet

The expression for the drag action on a droplet in Stokes flow region is given by

$$D = 6\pi\mu_0 U_\infty R \frac{1 + \frac{2\mu_0}{3\mu_d}}{1 + \frac{\mu_0}{\mu_d}}$$

where μ_0 the viscosity of the outer stream and μ_d is the viscosity of the droplet.

Comparisons of Different Drag Laws

Figures 4 and 5 compare various drag laws and their ranges of applicability. It is seen that the Stokes and Oseen drag are valid for very small Reynolds numbers. The empirical equation given by (35), however, covers a the entire range of Reynolds number up to about 1000.

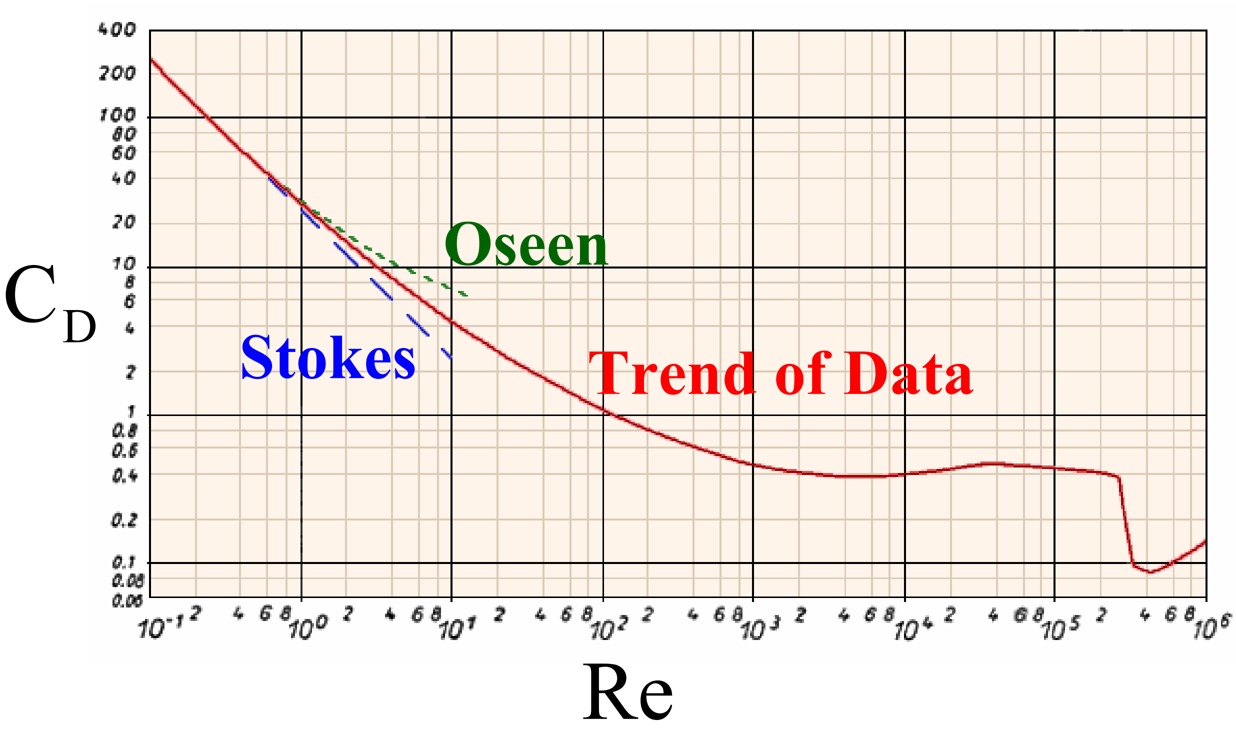


Figure 4. Predictions of various models for drag coefficient for a spherical particle.

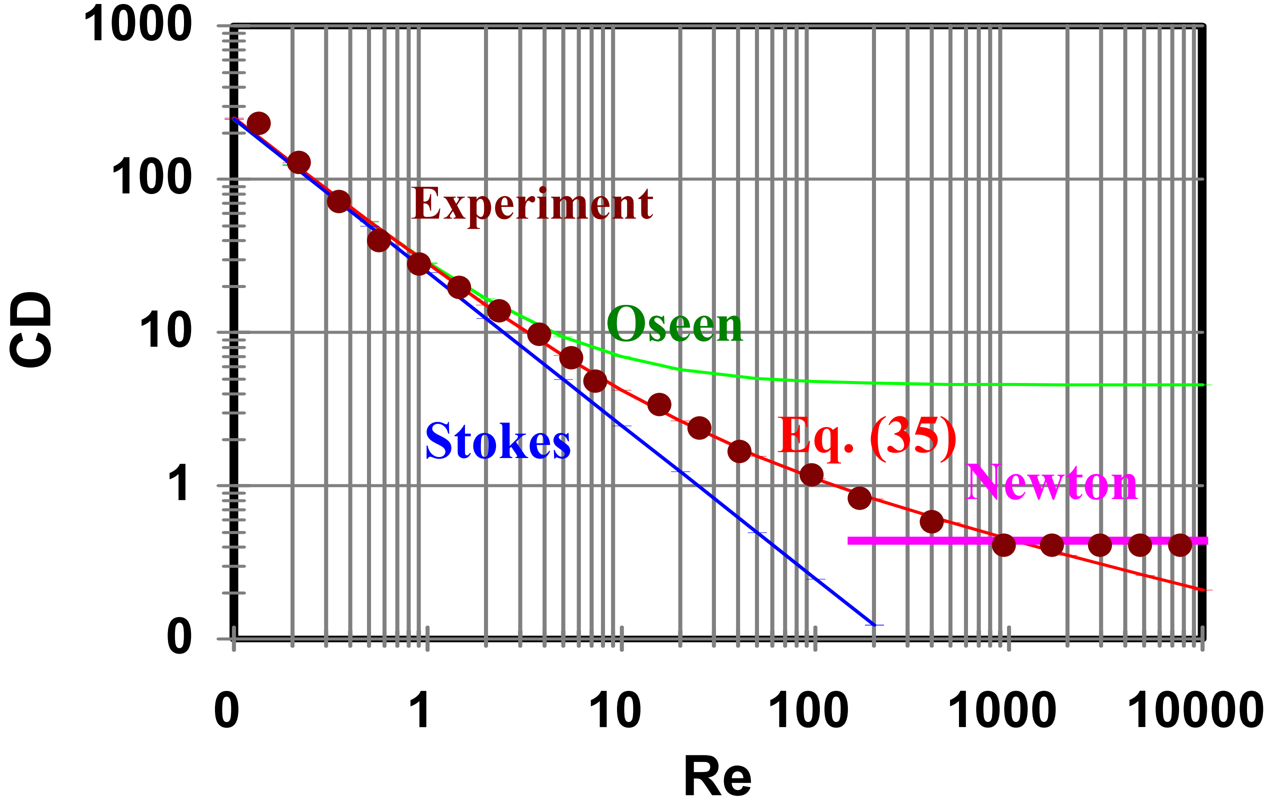


Figure 5. Predictions of various models for drag coefficient for a spherical particle.