

Review of Engineering Mathematics

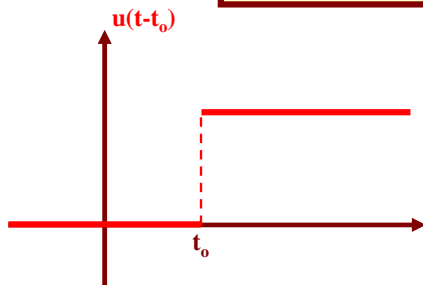
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- Special Functions
- Differential Equations
- Fourier Series and Transforms
- Probability and Random Processes
- Linear System Analysis

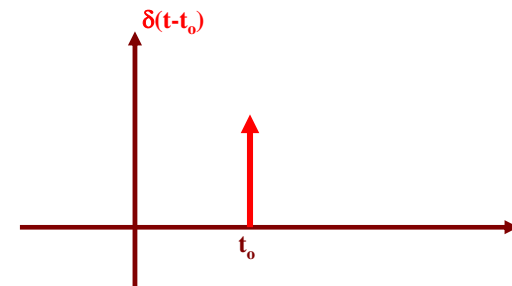
**Unit Step
Function**

$$u(t - t_0) = \begin{cases} 1 & t \geq t_0 \\ 0 & t < t_0 \end{cases}$$



**Dirac Delta
Function**

$$\delta(t - t_0) = \frac{du(t - t_0)}{dt}$$



Properties of Delta Function Clarkson University

$$\int_{-\infty}^{+\infty} \delta(t - t_0) dt = \int_{t_0 - \varepsilon}^{t_0 + \varepsilon} \delta(t - t_0) dt = 1$$

$$\int_{-\infty}^{+\infty} f(t) \delta(t - t_0) dt = \int_{t_0 - \varepsilon}^{t_0 + \varepsilon} f(t) \delta(t - t_0) dt = f(t_0)$$

$$\int_{-\infty}^t f(t_1) \delta(t_1 - t_0) dt_1 = f(t_0) u(t - t_0)$$

$$\delta[a(t - t_0)] = \frac{1}{|a|} \delta(t - t_0)$$

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Special Functions Clarkson University

Error Function

$$\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt$$

$$\operatorname{erfc}(x) = 1 - \operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_x^{\infty} e^{-t^2} dt$$

$$\operatorname{erf}(0) = 0 = \operatorname{erfc}(\infty)$$

$$\operatorname{erf}(-x) = -\operatorname{erf}(x)$$

Exponential Integrals

$$E_1(x) = \int_{-\infty}^x \frac{e^t}{t} dt$$

$$E_n(x) = \int_1^{\infty} \frac{e^{-xt}}{t} dt$$

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Differential Equations Clarkson University

Linear First-Order

$$\frac{dy}{dx} + P(x)y = Q(x)$$

$$y = ce^{-\int_0^x P(x_1) dx_1} + \int_0^x e^{-\int_{x_1}^x P(x_1) dx_2} Q(x_1) dx_1$$

$$\frac{dy}{dx} + by = Q(x) \implies y = \int_0^x e^{-b(x-x_1)} Q(x_1) dx_1$$

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Second-Order Equations Clarkson University

$$\frac{d^2y}{dx^2} + a \frac{dy}{dx} + by = 0$$

$$m^2 + am + b = 0 \quad \text{Solve for } \rightarrow m_1, m_2$$

$$m_2 \neq m_1 = \text{Real} \implies y = c_1 e^{m_1 x} + c_2 e^{m_2 x}$$

$$m_2 = m_1 = m \implies y = e^{mx} (c_1 + c_2 x)$$

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Second-Order Equations Clarkson University

$$m_1 = p + qi$$

$$p = -\frac{a}{2}$$

$$m_2 = p - qi$$

$$q = \sqrt{b - \frac{a^2}{4}}$$



$$y = e^{px} (c_1 \cos qx + c_2 \sin qx)$$

Second-Order Equations Clarkson University

Particular Solutions

$$\frac{d^2y}{dx^2} + a \frac{dy}{dx} + by = R(x)$$

$$y_p = \frac{e^{m_1x}}{m_1 - m_2} \int e^{-m_1x} R(x) dx + \frac{e^{m_2x}}{m_2 - m_1} \int e^{-m_2x} R(x) dx$$

$$y_p = xe^{mx} \int e^{-mx} R(x) dx - e^{mx} \int xe^{-mx} R(x) dx$$

$$y_p = \frac{e^{px} \sin qx}{q} \int e^{-px} R(x) \cos qx dx - \frac{e^{px} \cos qx}{q} \int e^{-px} R(x) \sin qx dx$$

Euler Equation Clarkson University

$$x^2 \frac{d^2y}{dx^2} + ax \frac{dy}{dx} + by = s(x)$$

Let $y = Ax^m \implies m(m-1) + am + b = 0$

$$y = A_1x^{m_1} + A_2x^{m_2}$$

Homogeneous Equations Clarkson University

$$\frac{dy}{dx} = F\left(\frac{y}{x}\right)$$

Let $v = \frac{y}{x} \implies \frac{dy}{dx} = x \frac{dv}{dx} + v$

$$\ln x = \int \frac{dv}{F(v) - v} + c$$

Exact Equations Clarkson University

$$M(x, y)dx + N(x, y)dy = 0$$

With

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} = \frac{\partial^2 \phi(x, y)}{\partial x \partial y}$$

$$M = \frac{\partial \phi}{\partial x}$$

$$N = \frac{\partial \phi}{\partial y}$$

$$\phi(x, y) = \text{const}$$

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Bessel's Equation Clarkson University

$$x^2 \frac{d^2 y}{dx^2} + x \frac{dy}{dx} + (\beta^2 x^2 - n^2)y = 0$$

Solutions

$$y = C_1 J_n(\beta x) + C_2 Y_n(\beta x)$$

Bessel Functions

$$J_n(\beta x)$$

$$Y_n(\beta x)$$

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Fourier Series Clarkson University

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi x}{L} + b_n \sin \frac{n\pi x}{L} \right)$$

$$a_n = \frac{1}{L} \int_{-L}^L f(x) \cos \frac{n\pi x}{L} dx$$

$$b_n = \frac{1}{L} \int_{-L}^L f(x) \sin \frac{n\pi x}{L} dx$$

Fourier Cosine Series

$$f(x) = f(-x)$$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{L}$$

$$a_n = \frac{2}{L} \int_0^L f(x) \cos \frac{n\pi x}{L} dx$$

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Fourier Sine Series Clarkson University

$$f(x) = -f(-x)$$

$$f(x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{L}$$

$$b_n = \frac{2}{L} \int_0^L f(x) \sin \frac{n\pi x}{L} dx$$

Fourier Exponential Series

$$\omega_n = \frac{n\pi}{L}$$

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{i\omega_n x}$$

$$c_n = \frac{1}{2L} \int_{-L}^L f(x) e^{-i\omega_n x} dx$$

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From Fourier Series to Fourier Transforms

FES

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{\frac{i n \pi x}{L}}$$

$$-L < x < L$$

$$c_n = \frac{1}{2L} \int_{-L}^L e^{-\frac{i n \pi x}{L}} f(x) dx$$

Replacing for c_n

$$f(x) = \frac{1}{2L} \sum_{n=-\infty}^{\infty} \int_{-L}^L e^{i \omega_n (x-x')} f(x') dx'$$

$$\omega_n = \frac{n\pi}{L}$$

$$\Delta\omega = \frac{\pi}{L}$$

$$L \rightarrow \infty$$

$$\sum g_n \Delta\omega = \int g d\omega$$

Fourier Transforms

Fourier Integral Representation

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} e^{i\omega(x-x')} f(x') dx' d\omega$$

Fourier Transform (Exponential)

$$\bar{f}(\omega) = \int_{-\infty}^{+\infty} e^{-i\omega x'} f(x') dx'$$

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{i\omega x} \bar{f}(\omega) d\omega$$

Fourier Cosine and Sine Transforms

$$\bar{f}_c(\omega) = \int_0^{\infty} \cos \omega x' f(x') dx'$$

Fourier Cosine Transform

$$f(x) = \frac{2}{\pi} \int_0^{\infty} \cos \omega x \bar{f}_c(\omega) d\omega$$

$$\bar{f}_s(\omega) = \int_0^{\infty} \sin \omega x' f(x') dx'$$

Fourier Sine Transform

$$f(x) = \frac{2}{\pi} \int_0^{\infty} \sin \omega x \bar{f}_s(\omega) d\omega$$

Fourier Transform of Derivatives

$$\mathfrak{T} \left\{ \frac{df}{dx} \right\} = \int_{-\infty}^{+\infty} e^{-i\omega x} \frac{df(x)}{dx} dx = i\omega \bar{f}(\omega)$$

$$\mathfrak{T} \left\{ \frac{d^2 f}{dx^2} \right\} = -\omega^2 \bar{f}(\omega) \quad \mathfrak{T} \left\{ \frac{d^n f}{dx^n} \right\} = (i\omega)^n \bar{f}(\omega)$$

Example

$$\frac{d^2f}{dx^2} + a \frac{df}{dx} + bf = \delta(x - x_0)$$

$$-\infty < x < +\infty$$

Taking Fourier Transform

$$-\omega^2 \bar{f}(\omega) + ai\omega \bar{f}(\omega) + b\bar{f}(\omega) = e^{-i\omega x_0}$$

$$\bar{f}(\omega) = \frac{e^{-i\omega x_0}}{b - \omega^2 + ia\omega}$$

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \frac{e^{i\omega(x-x_0)}}{b - \omega^2 + ia\omega} d\omega$$

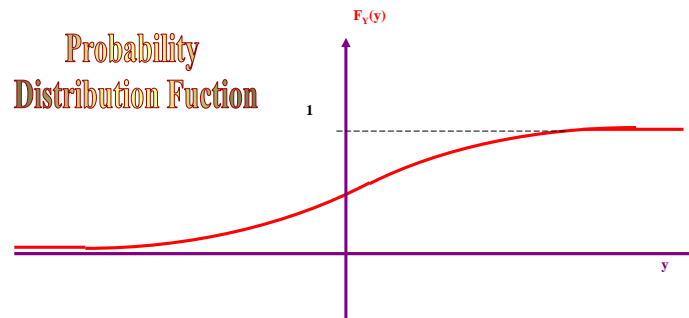
Table of Fourier Transform

$f(x)$	$\bar{f}(\omega)$
$f_1(x + x_0)$	$e^{i\omega x_0} \bar{f}(\omega)$
$\delta(x - x_0)$	$e^{-i\omega x_0}$
$e^{-\alpha x }$	$\frac{2\alpha}{\omega^2 + \alpha^2}$
$f_1(x) * f_2(x) = \int_{-\infty}^{+\infty} f_1(\xi) f_2(x - \xi) d\xi$	$\bar{f}_1(\omega) \bar{f}_2(\omega)$
$\cos \omega_0 x$	$\pi[\delta(\omega - \omega_0) + \delta(\omega + \omega_0)]$
$e^{-\alpha x } \cos \beta x$	$\frac{2\alpha(\omega^2 + \alpha^2 + \beta^2)}{(\omega^2 - \beta^2 - \alpha^2)^2 + 4\alpha^2\omega^2}$
$e^{-\alpha x } \left[\cos \beta x + \frac{\alpha}{\beta} \sin \beta x \right]$	$\frac{4\alpha(\alpha^2 + \beta^2)}{(\omega^2 - \beta^2 - \alpha^2)^2 + 4\alpha^2\omega^2}$

Table of Fourier Transform

$f(x)$	$\bar{f}(\omega)$
$e^{-\alpha^2 x^2} \cos \beta x$	$\frac{\sqrt{\pi}}{2\alpha} \left[\exp\left\{-\frac{(\omega + \beta)^2}{4\alpha^2}\right\} + \exp\left\{-\frac{(\omega - \beta)^2}{4\alpha^2}\right\} \right]$
$e^{-\alpha^2 x^2}$	$\frac{\sqrt{\pi}}{\alpha} \exp\left\{-\frac{\omega^2}{4\alpha^2}\right\}$
$\frac{d^n}{dx^n} \delta(x)$	$(i\omega)^n$
$J_0(x)$	$\begin{cases} \frac{2}{\sqrt{1-\omega^2}} & \omega < 1 \\ 0 & \text{elsewhere} \end{cases}$

Probability and Random Processes

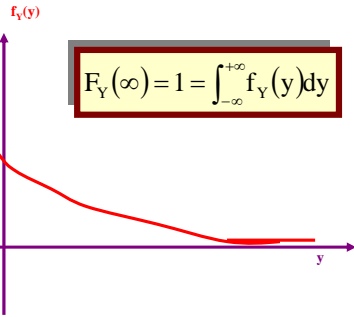


$$F_Y(y) = P\{Y \leq y\}$$

$$0 \leq F_Y(y) \leq 1$$

Probability Density Function Clarkson University

$$f_Y(y) = \frac{dF_Y(y)}{dy}$$



$$F_Y(\infty) = 1 = \int_{-\infty}^{+\infty} f_Y(y) dy$$

$$P\{y_1 < Y \leq y_2\} = \int_{y_1}^{y_2} f_Y(y) dy = F_Y(y_2) - F_Y(y_1)$$

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Probability and Random Processes Clarkson University

Expected Value

$$E\{Y\} = \bar{Y} = \int_{-\infty}^{+\infty} y f_Y(y) dy$$

$$E\{g(Y)\} = \overline{g(Y)} = \int_{-\infty}^{+\infty} g(y) f_Y(y) dy$$

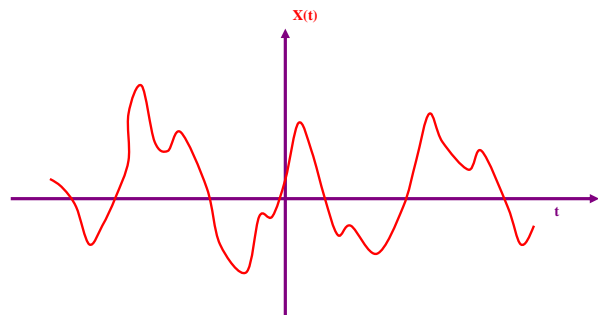
Variance

$$\sigma_Y^2 = E\{(Y - \bar{Y})^2\} = E\{Y^2\} - \bar{Y}^2$$

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Stochastic Processes Clarkson University



$$E\{X(t)\} = \int_{-\infty}^{+\infty} x f_X(x, t) dx$$

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Stochastic Processes Clarkson University

Time Averaging

$$\bar{X}(t) = \frac{1}{T} \int_0^T X(t) dt \approx E\{X(t)\}$$

Autocorrelation

$$R_{xx}(\tau) = E\{X(t+\tau)X(t)\} = \frac{1}{T} \int_0^T X(t+\tau)X(t) dt$$

$$R_{xx}(0) = E\{X^2(t)\} = \overline{X^2(t)}$$

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Energy Spectrum Clarkson University

$$S_{xx}(\omega) = \int_{-\infty}^{+\infty} e^{-i\omega\tau} R_{xx}(\tau) d\tau$$

$$R_{xx}(\tau) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{i\omega\tau} S_{xx}(\omega) d\omega$$

$$S_{xx}(\omega) = \frac{1}{T} |\tilde{X}(\omega)|^2$$

$$\tilde{X}(\omega) = \int_{-\infty}^{+\infty} e^{-i\omega t} X(t) dt$$

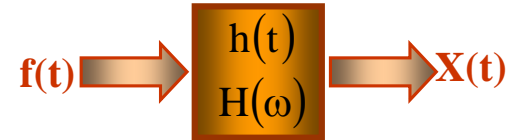
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Linear Systems Analysis Clarkson University

h(t)=Impulse Response

H(ω)=System Function



$$X(t) = \int_0^t h(t-\tau) f(\tau) d\tau$$

$$H(\omega) = \int_{-\infty}^{+\infty} e^{-i\omega t} h(t) dt$$

$$X(t) = \int_{-\infty}^{+\infty} h(t-\tau) f(\tau) d\tau = h(t) * f(t)$$

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Linear Systems Analysis Clarkson University

Fourier Transform

$$\tilde{x}(\omega) = H(\omega) \tilde{f}(\omega)$$

$$S_{xx}(\omega) = \frac{1}{T} |\tilde{x}(\omega)|^2 = \frac{1}{T} |H(\omega)|^2 |\tilde{f}(\omega)|^2 = |H(\omega)|^2 S_{ff}(\omega)$$

Spectral Relationship

$$S_{xx}(\omega) = |H(\omega)|^2 S_{ff}(\omega)$$

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Linear Systems Analysis - Impulse Response Clarkson University

$$\dot{x} + \alpha x = f(t) \implies h(t) = e^{-\alpha t}$$

$$\ddot{x} + 2\zeta\omega_0\dot{x} + \omega_0^2 x = f(t)$$

$$h(t) = \frac{1}{\omega_d} e^{-\zeta\omega_0 t} \sin \omega_d t \quad \omega_d = \omega_0 \sqrt{1 - \zeta^2}$$

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Numerical Solution of Differential Equations

$$\dot{x} + \alpha x = n(t)$$

Finite difference

$$\frac{x_{i+1} - x_i}{\Delta t} + \alpha x_{i+1} = n_i$$

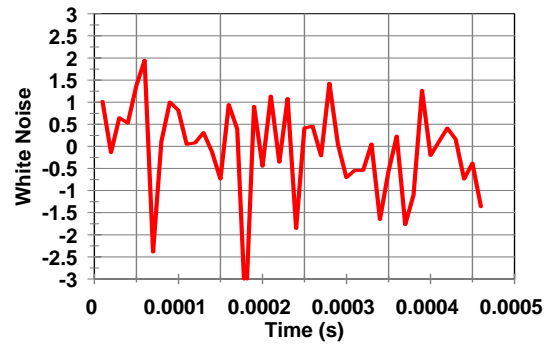
$$x_{i+1} = \frac{1}{1 + \alpha \Delta t} x_i + \frac{\Delta t}{1 + \alpha \Delta t} n_i$$

$$x_i = x(t_i)$$

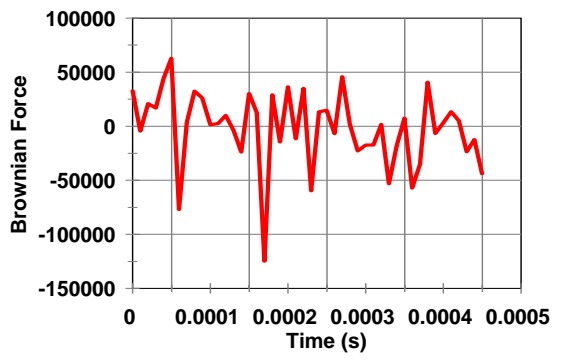
$$x_{i+1} = x_i + \Delta t n_i \quad \text{for } \alpha \Delta t \ll 1$$

$$x_{i+1} = \frac{1}{\alpha \Delta t} x_i + \frac{1}{\alpha} n_i \quad \text{for } \alpha \Delta t \gg 1$$

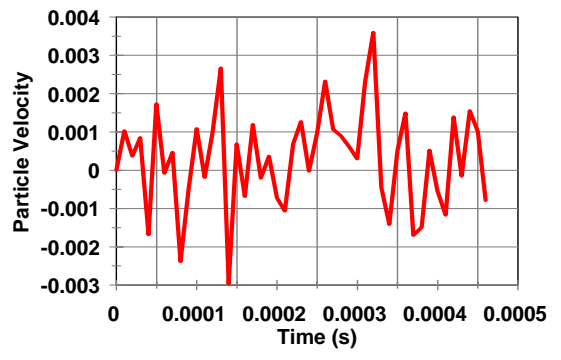
White Noise



Brownian Force

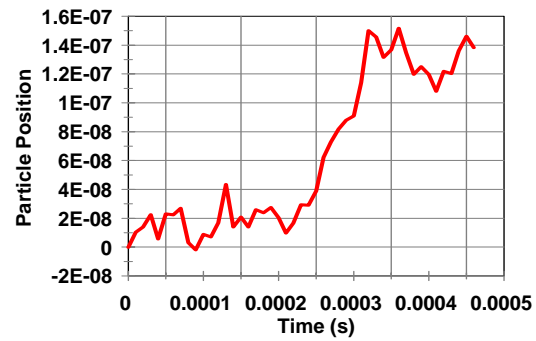


Particle Velocity



Particle Position

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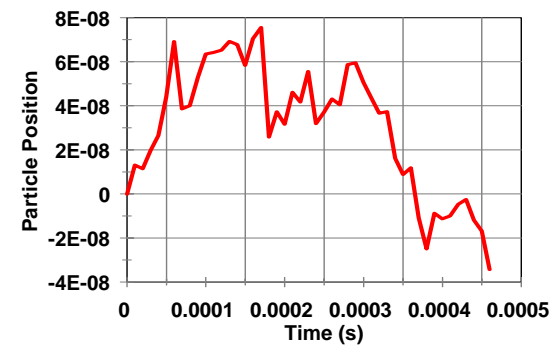


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Particle Position

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