ME529 **Homework 3**

1. Show that the characteristic function of a random variable X must be positive-definite. i.e.,

$$\sum_{m=1}^{n}\sum_{k=1}^{n}\Phi(\omega_m-\omega_k)a_m a_k^* \geq 0$$

for any complex number a_j.

2. i) Find $F_Y(y)$ in terms of $F_X(x)$ directly for $Y = 1/X$

   ii) For X uniformly distributed between 0,1, i.e.,

$$F_X(x) = \begin{cases} 0 & x \leq 0 \\ x & 0 < x \leq 1 \\ 1 & x > 1 \end{cases}$$

   Find . $F_Y(y)$

   iii) If X is normally distributed ,

$$f_X(x) = \frac{1}{\sqrt{2\pi\sigma}} e^{-x^2/2\sigma}$$

   find the density of Y.

3. For $Y = 1/|X|$, find $F_Y(y)$ directly in terms of $F_X(x)$, find the corresponding density function, find the density function $f_Y(y)$ directly from $f_X(x)$.

4. For $W = X - Y$, find $F_W(w)$ directly. Find $f_W(w)$ with the use of an auxiliary variable. If X and Y are independent random variables with respective densities $\alpha e^{-\alpha x}U(x)$ and $\beta e^{-\beta y}U(y)$, find the density W.

5. Find the joint density of Z and W with $Z = X + Y, W = X/(X+Y)$. If $f_X(x) = e^{-x}U(x), f_Y(y) = e^{-y}U(y)$ and X and Y are independent random variable. Show that Z and W are also independent random variable and $f_Z(z) = ze^{-z}U(z)$, W is uniformly distributed in the interval (0,1).

6. $\overline{X} = \frac{1}{n}\sum_{j=1}^{n} X_j$ with $X_j, j = 1, \cdots n$ being independent random variable with $f_{X_i}(x_i) = \frac{\alpha / \pi}{\alpha^2 + x_i^2}$, show that $f_{\overline{X}}(\overline{x})$ is also Cauchy. Discuss why the central limit theorem does not hold for this sequence.

7. Consider the die rolling experiment. We define a random variable $X(\zeta_j) = j$, j being the side shown. Assume that $P(\zeta_j) = 1/6$ for $j = 1, 2, \cdots 6$
   a) Determine E(X).
   b) Find $\phi_X(\omega)$

8. The joint density function of X and Y is given as

$$f_{XY}(x, y) = \begin{cases} \frac{1}{ab} & 0 \le x \le a, 0 \le y \le b \\ 0 & otherwise \end{cases}$$

   a) Determine E(X), E(XY) and the joint characteristic function $\Phi_{XY}\left(\omega_1, \omega_2\right)$.
   b) Find $P(X \le Y)$.

9. The density of a random variable X is given by

$$f_X(x) = \frac{\alpha}{2} e^{-\alpha|x|}$$

   Let $Y = \sqrt{X} U(X)$
   a) Determine $f_Y(y)$.
   b) Find E(Y) and sketch $F_Y(y)$

10. Two independent random variables X and Y are uniformly distributed on (0,1). Given that

$$Z = \sqrt{-2 \ln X} Cos 2\pi Y$$

$$W = \sqrt{-2 \ln X} Sin 2\pi Y$$

   Find the joint density of Z and W.