The Karhunen-Loeve Orthogonal Expansion

Suppose $\varphi_n(t)$ are a set of orthonormal basis in the interval (0,T). A function *X(t)* (deterministic or random) may be expanded as

$$X(t) = \sum c_n \varphi_n(t), \qquad 0 < t < T \ , \tag{1}$$

where the coefficient c_n are given by

$$c_n = \int_0^T X(t)\varphi_n(t)dt \, . \tag{2}$$

Note that the property

$$\int_0^T \varphi_n(t)\varphi_m^*(t)dt = \delta_{nm} \tag{3}$$

was used in the derivation of (2).

When *X(t)* is a random function, the coefficients c_n become random coefficients) variables. In the following, assume $\mathrm{E}\{\mathrm{X}\} = 0$.

Theorem: In the expansion (1), the coefficients c_n become uncorrelated (orthogonal) random variables if and only if $\varphi_n(t)$ are the eigenfunction of the following Fredholm's integral equation:

$$\int_0^T R_{xx}(t_1, t_2)\varphi_n(t_2) = \lambda_n \varphi_n(t_1). \tag{4}$$

In this case,

$$E\left\{|c_n|^2\right\} = \lambda_n \, . \tag{5}$$

Proof: From (1) and (2), it follows that

$$E\{x^*(t_1)c_n\} = E\left\{|c_n|^2\right\}\varphi_n = \int_0^T R_{xx}(t_1, t_2)\varphi_n(t_2)dt_2 \ , \tag{6}$$

where

$$E\{c_n c_m^*\} = E\left\{|c_n|^2\right\}\delta_{nm} \tag{7}$$

is used. Thus $\lambda_n = E\{|c_n|^2\}$ in Equation (4).

It may be also shown that the Karhunen-Loeve ($\mathrm{K-L}$) expansion converges in mean-square sense, i.e.

$$E\left\{\left[X(t)-\sum_n c_n \varphi_n(t)\right]^2\right\}=0\,. \tag{8}$$

(See Papoulis page 304 for details.) It may also be easily shown that

$$R_{xx}(t_1,t_2)=\sum_n \lambda_n |\varphi_n|^2\,. \tag{9}$$

Stationary and Periodic Processes

If $X(t)$ is stationary, then $R_{xx}=R_{xx}(t_1-t_2)$. If in addition, $\mathrm{X}(\mathrm{t})$ is also periodic in the mean-square sense, then

$$\varphi_n(t)=\frac{1}{\sqrt{T}}e^{in\omega_0 t}\,,\qquad \omega_0=\frac{2\pi}{T}\,. \tag{10}$$

The $\mathrm{K-L}$ expansion for $x(t)$ and R_{xx} are given as

$$x(t)=\sum_{-\infty}^{+\infty}\frac{c_n}{\sqrt{T}}e^{in\omega_o t}\,,\qquad E\{|c_n|^2\}=\lambda_n \tag{11}$$

$$R_{xx}(t_1,t_2)=\frac{1}{T}\sum_{-\infty}^{+\infty}\lambda_n e^{in\omega_0(t_1-t_2)} \tag{12}$$

The power spectrum of $X(t)$ is then given by

$$S_{xx}(\omega)=\frac{1}{T}\sum_{-\infty}^{+\infty}\lambda_n \delta(\omega-n\omega_0)\,. \tag{13}$$

Furthermore,

$$E\{X^2(t)\}=\frac{1}{T}\sum_{-\infty}^{+\infty}\lambda_n\,. \tag{14}$$

Stationary Nonperiodic Processes

These may be considered to have infinite period. One may write

$$X(t)=\int_{-\infty}^{+\infty} e^{i\omega t} n(\omega)\sqrt{S(\omega)}d\omega\,, \tag{15}$$

where $n(\omega)$ is a white noise in frequency space with

$$E\{n(\omega_1)n(\omega_2)\}=\delta(\omega_1-\omega_2) \tag{16}$$

Autocorrelation of $X(t)$ then is given as

$$R_{xx}(t_1,t_2)=\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} e^{i\omega_1 t_1-i\omega_2 t_2}\delta(\omega_1-\omega_2)\sqrt{s(\omega_1)s(\omega_2)}d\omega_1 d\omega_2$$

Hence,

$$R_{xx}(t_1-t_2)=\int_{-\infty}^{+\infty} e^{-i\omega(t_2-t_1)}s(\omega)d\omega\,, \tag{17}$$

as is expected.

Responses of a Linear System to White Excitations

Consider a linear system

$$L_t X(t)=n(t),\quad X(0)=X'(0)=\ldots=0\,, \tag{18}$$

with

$$R_{nn}(t_1,t_2)=2\pi S_0\delta(t_1-t_2). \tag{19}$$

Solution to (18) is given by

$$X(t)=\int_0^t h(t-\tau)n(\tau)d\tau\,, \tag{20}$$

where $h(t)$ is the impulse response. i.e.,

$$L_t h(t)=\delta(t). \tag{21}$$

Multiplying (20) by $X(t_2)$ and apply L_t, after averaging one finds

$$L_t R_{xx}(t,t_2) = L_t \int_0^t h(t-\tau) E\{n(\tau) X(t_2)\} d\tau$$

or

$$L_t R_{xx}(t,t_2) = E\{n(t) X(t_2)\} = 2\pi S_0' h(t_2 - t). \tag{22}$$

Restating equation (4) as

$$\int_o^T R_{xx}(t,t_2)\varphi(t_2) dt_2 = \lambda \varphi(t) \tag{23}$$

and applying L_t and using (22), it follows that

$$\int_0^T 2\pi S_0 h(t_2 - t)\varphi(t_2) dt_2 = \lambda L_t \phi(t) \tag{24}$$

Operating on (24) with L_{-t} (same as L_t with t being replaced by $-$t) and using (21), it follows that

$$\lambda L_{-t} L_t \phi(t) = \int_0^T 2\pi S_0 \delta(t_2 - t)\phi(t_2) dt_2 = 2\pi S_0 \varphi(t) \tag{25}$$

Equation (25) is a differential equation for evaluating the eigenfunction $\varphi_n(t)$ and eigenvalues λ_n. It may be shown that if L_t involves derivations of the order N, then the following boundary conditions may be used:

$$\varphi^{(i)}(0) = 0 \qquad \text{for } i = 0,1,\ldots,N-1 \tag{26}$$

$$L_t \varphi^{(i)}(t)\big|_{t=T} = 0 \qquad \text{for } i = 0,1,\ldots,N-1 \tag{27}$$

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On the Convergence of Karhunen-Loeve Series Expansion for a Brownian Particle

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A linear Langevin equation for the velocity of a Brownian particle is considered. The equation of motion is solved and the Karhunen-Loeve expansion for the particle velocity is derived. The mean-square velocity as obtained by the truncated Karhunen-Loeve expansion is compared with the exact solution. It is shown, as the number of terms in the series increases, the result approaches that of the exact solution asymptotically.

Introduction

Brownian motion was first observed by Robert Brown in 1827 while studying pollen particles suspended in liquid, and Brownian diffusivity was first estimated by Einstein (1903). An extensive exposition of the theory of Brownian motion was provided by Chandrasekhar (1943).

Use of the Karhunen-Loeve (KL) expansion (Loeve, 1955) for representing random data has attracted considerable attention in the field of turbulence (Lumley, 1967) and other areas (Lin and Yong, 1986). Here, the Karhunen-Loeve expansion for a Brownian particle is considered and analytical expressions for orthogonal basis are derived. The particle velocity response statistics as evaluated from the truncated series are compared with the exact values and the convergence of the KL series is discussed.

Analysis

Equation of Motion. The linear Langevin equation for the velocity of a Brownian particle is given as

$$\frac{du}{dt} + \beta u = n(t) \tag{1}$$

where

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$$\beta = \frac{3\pi\mu d}{C_c m} \tag{2}$$

and $n(t)$ is a zero-mean Gaussian white noise process with a constant spectral intensity, S_o, given by

$$S_o = \frac{216\nu k T\rho}{\pi^2 d^5 \rho_p^2 C_c}. \tag{3}$$

Here, μ is the kinematic viscosity, d is the particle diameter, C_c is the Cunningham correction factor, m is the mass of the particle, ν is the kinematic viscosity, k is the Boltzmann constant, T is the temperature, ρ is the fluid density, and ρ_p is the particle density. A white noise process may be formally defined as the derivative of a Wiener process (Papoulis, 1984). A digital simulation procedure for generating white noise process corresponding to molecular agitation was described by Ounis et al. (1991).

Assuming that the motion starts from rest,

$$u(0) = 0. \tag{4}$$

Equation (4) is the initial condition for particle velocity.

Karhunen-Loeve Expansion. According to the Karhunen-Loeve Theorem (Loeve, 1955), the random velocity has a series expansion of the form

$$u(t) = \sum_{n=1}^{\infty} C_n \Phi_n(t) \tag{5}$$

where $\Phi_n(t)$ are the KL orthonormal basis and C_n are independent random coefficients. The KL basis are the eigenfunctions of the Fredholm equation given by

$$\int_0^T R_{uu}(t_1,t_2)\Phi_n(t_2)dt_2 = \lambda_n \Phi_n(t_1). \tag{6}$$

Here the kernel $R_{uu}(t_1, t_2)$ is the particle velocity autocorrelation function, and eigenvalues $\lambda_n = \langle |C_n|^2 \rangle$, with "$\langle\,\rangle$" denoting the expected value (ensemble average) and T, is a specified time duration.

Following the procedure outlined by Lin and Yong (1986), Eq. (6) may be restated as

$$L_{-t} L_t \Phi_n(t) = \frac{2\pi S_o}{\lambda_n}\Phi_n(t) \tag{7}$$

where

$$L_t = \frac{d}{dt} + \beta, \quad L_{-t} = -\frac{d}{dt} + \beta. \tag{8}$$

The required boundary conditions are

$$\Phi_n(0) = 0, \quad L_t \Phi_n(T) = 0. \tag{9}$$

The eigenfunctions for the boundary value problem, (7)–(9), are given by

$$\Phi_n(t) = A_n \sin(\xi_n t) \tag{10}$$

where

$$\xi_n = \sqrt{\frac{2\pi S_o}{\lambda_n} - \beta^2} \tag{11}$$

are solutions to the transcendental equation

$$\tan(\xi_n T) = \frac{-\xi_n}{\beta}. \tag{12}$$

The corresponding eigenvalues are

$$\lambda_n = \frac{2\pi S_o}{\beta^2 + \xi_n^2}. \tag{13}$$

Using the normality condition,

Table 1 Listing of first nine eigenvalues for $\beta T = 5$

n	1	2	3	4	5	6	7	8	9
$\xi_n T$	2.654	5.454	8.391	11.409	14.47	17.556	20.657	23.769	26.887
$\frac{100\lambda_n}{2\pi S_o T}$	3.12	1.83	1.05	0.644	0.427	0.3	0.221	0.17	0.134

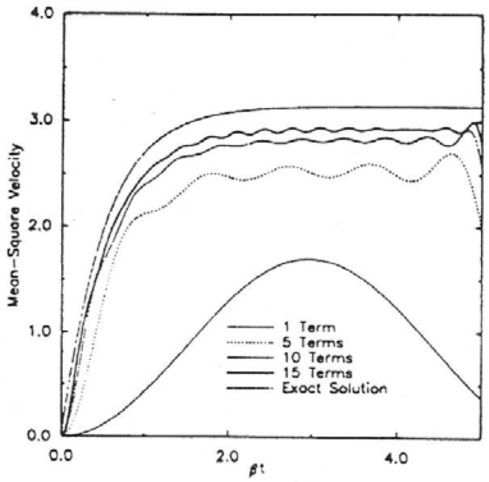

Fig. 1 Comparison of the mean-square velocities for the truncated Karhunen-Loeve series

$$\int_0^T \Phi_n(t)^2 dt = 1, \tag{14}$$

the coefficients A_n in Eq. (10) are given as

$$A_n = \sqrt{\frac{2\xi_n}{\xi_n T - \sin(\xi_n T)}}. \tag{15}$$

The mean-square velocity associated with the KL series is given by

$$\langle u^2(t)\rangle = \sum_n \lambda_n |\Phi_n(t)|^2 \tag{16}$$

and the exact transient mean-square velocity response as obtained by use of the impulse response method is

$$\langle u^2(t)\rangle = \frac{\pi S_o}{\beta}(1 - e^{-2\beta t}). \tag{17}$$

Results

For a nondimensional time duration of $\beta T = 5$, Table 1 provides a listing of the first nine values of $\xi_n T$ and λ_n. The weightings of different modes in Eq. (15) which correspond to eigenvalues λ_n can be clearly seen from this table. It is observed that λ_{n-1} is roughly about 80 percent of λ_n for higher modes. Figure 1 compares the dimensionless mean-square velocity responses, $\beta\langle u^2\rangle/S_o$, as obtained by the truncated KL expansion with the exact solution given by (17). The gradual convergence of the series solution to the exact mean-square response is clearly observed from this figure. Figure 2 shows

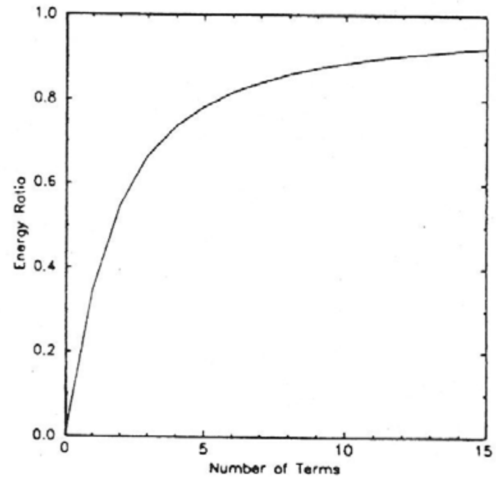

Fig. 2 Variation of energy ratio with order of truncation

the energy ratio for various orders of truncation of the KL series. Here the energy ratio is defined as the ratio of area under the mean-squared response curve as obtained by the truncated KL series to that of the exact one. From Fig. 2 it is observed that the first few terms of the KL series capture most of the energy. However, the convergence is asymptotic and a large number of terms are needed to recover the exact result.

Conclusion

For a finite time duration, the exact Karhunen-Loeve orthogonal basis for Brownian particles are derived. The mean-square velocities as evaluated from the truncated KL series expansion are compared with the exact one. It is shown that the first few terms of the series contains a substantial fraction of the energy of the response. However, for a high resolution description, consideration of a large number of terms are required.

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