Equivalent Linearization Technique

Consider a non-linear system given as

$$\ddot{X} + g\left(X, \dot{X}\right) = f(t), \tag{1}$$

where f(t) is a random process and $g\left(X, \dot{X}\right)$ is an arbitrary function of X and $\dot{X}$. We assume that Equation (1) may be replaced by its equivalent linear system, which is given as

$$\ddot{X} + \beta_e \dot{X} + \omega_e^2 X = f(t), \tag{2}$$

where β_e is the equivalent damping and ω_e is the equivalent natural frequency.

The mean-square error for replacing Equation (1) by (2) is given as

$$E\left\{(error)^2\right\} = E\left\{e^2\right\} = E\left\{\left(\beta_e \dot{X} + \omega_e^2 X - g\left(X, \dot{X}\right)\right)^2\right\}. \tag{3}$$

The equivalent parameters are selected in such a way that the mean-square error given by (3) is a minimum. That is,

$$\frac{\partial}{\partial \beta_e} E\left\{e^2\right\} = 0 = 2E\left\{\dot{X}\left[\beta_e \dot{X} + \omega_e^2 X - g\left(X, \dot{X}\right)\right]\right\}, \tag{4}$$

$$\frac{\partial}{\partial\left(\omega_e^2\right)} E\left\{e^2\right\} = 0 = 2E\left\{X\left[\beta_e \dot{X} + \omega_e^2 X - g\left(X, \dot{X}\right)\right]\right\}. \tag{5}$$

From (4) and (5) it follows that

$$\beta_e E\left\{\dot{X}^2\right\} + \omega_e^2 E\left\{X\dot{X}\right\} = E\left\{\dot{X} g\left(X, \dot{X}\right)\right\}, \tag{6}$$

$$\omega_e^2 E\left\{X^2\right\} + \beta_e E\left\{X\dot{X}\right\} = E\left\{X g\left(X, \dot{X}\right)\right\}. \tag{7}$$

Solving for the equivalent parameters, we find

$$\omega_e^2 = \frac{E\left\{\dot{X}^2\right\}E\left\{Xg\right\} - E\left\{X\dot{X}\right\}E\left\{\dot{X}g\right\}}{E\left\{X^2\right\}E\left\{\dot{X}^2\right\} - \left(E\left\{X\dot{X}\right\}\right)^2}, \tag{8}$$

and

$$\beta_e = \frac{E\{X^2\}E\{\dot{X}g\} - E\{X\dot{X}\}E\{Xg\}}{E\{X^2\}E\{\dot{X}^2\} - \left(E\{X\dot{X}\}\right)^2}. \tag{9}$$

To evaluate the moments on the right hand sides of (8) and (9), the joint density of X and $\dot{X}$ is needed. When the excitation is a normal process, the common procedure is to assume that the response is also a normal process.

Stationary Response

For stationary response analysis, $E\{X\dot{X}\} = 0$, Equations (8) and (9) may then be restated as

$$\omega_e^2 = \frac{E\{Xg\}}{E\{X^2\}}, \qquad \beta e = \frac{E\{\dot{X}g\}}{E\{\dot{X}^2\}}.$$

(10)

From the fact that

$$\frac{\partial^2 E\{e^2\}}{\partial\left(\omega_e^2\right)^2} = 2E\{X^2\}, \quad \frac{\partial^2 E\{X^2\}}{\partial \beta_e^2} = 2E\{\dot{X}^2\}, \quad \frac{\partial^2 E\{e^2\}}{\partial \beta_e \partial\left(\omega_e^2\right)} = 2E\{X\dot{X}\}, \tag{11}$$

and

$$E\{X^2\}E\{\dot{X}^2\} - E\{X\dot{X}\} \geq 0, \tag{12}$$

it follows that the solutions given by (8) – (10) are a minimum and the mean square error is minimized.

Quasi-Gaussian Processes

For Gaussian processes, it may be shown that

$$\omega_e^2 = E\left\{\frac{\partial g}{\partial X}\right\}, \qquad \beta e = E\left\{\frac{\partial g}{\partial \dot{X}}\right\}. \tag{13}$$