Perturbation Techniques

Consider a general non-linear system subjected to an arbitrary random excitation Y(t). That is,

$$\ddot{X} + 2\beta\dot{X}(t) + \omega_0^2\left[X(t) + \varepsilon g\left(X, \dot{X}\right)\right] = Y(t), \tag{1}$$

with ε being a small parameter. We assume the solution to Equation (1) can be expanded in terms of powers of ε. i.e.,

$$X(t) = X_0(t) + \varepsilon X_1(t) + \varepsilon^2 X_2(t) + \ldots \tag{2}$$

Substituting Equation (2) in (1) and setting the coefficients of the various powers of ε equal to zero, we find

$$\ddot{X}_0 + 2\beta\dot{X}_0 + \omega_0^2 X_0 = Y(t), \tag{3}$$

$$\ddot{X}_1 + 2\beta\dot{X}_1 + \omega_0^2 X_1 = -\omega_0^2 g\left(X_0, \dot{X}_0\right), \tag{4}$$

$$\ddot{X}_2 + 2\beta\dot{X}_2 + \omega_0^2 X_2 = -\omega_0^2\left[\frac{\partial g\left(X_0, \dot{X}_0\right)}{\partial X_0} X_1 + \frac{\partial g\left(X_0, \dot{X}_0\right)}{\partial \dot{X}_0}\dot{X}_1\right]. \tag{5}$$

Note that we used

$$g\left(X, \dot{X}\right) = g\left(X_0, \dot{X}_0\right) + \varepsilon\left[\frac{\partial g}{\partial X_0} X_1 + \frac{\partial g}{\partial \dot{X}_0}\dot{X}_1\right] + \varepsilon^2[\ldots]. \tag{6}$$

Now Equations (3) to (5) are linear equations and can be solved. For instance with initial conditions,

$$X(0) = \dot{X}(0) = 0\,, \tag{7}$$

we find

$$X_0(t) = \int_0^t h(t-\tau)Y(\tau)d\tau\,, \tag{8}$$

$$X_1(t) = -\omega^2\int_0^t h(t-\tau)g\left[X_0(\tau), \dot{X}_0(\tau)\right]d\tau\,. \tag{9}$$

Here, the impulse response function h(t) is given by

$$h(t) = \frac{1}{\Omega_0} e^{-\beta t} \sin \Omega_0 t, \qquad \Omega_0^2 = \omega_0^2 - \beta^2 \tag{10}$$

The statistics of X(t) may be determined from Equation (2). These are

$$E\{X(t)\} = E\{X_0(t)\} + \varepsilon E\{X_1(t)\} + \dots \tag{11}$$

$$E\{X^2(t)\} = E\{X_0^2(t)\} + 2\varepsilon E\{X_0(t) X_1(t)\} + \dots \tag{12}$$

$$R_{XX}(t_1, t_2) = E\{X_0(t_1) X_0(t_2)\} + \varepsilon\left[E\{X_0(t_1) X_1(t_2)\} + E\{X_0(t_2) X_1(t_1)\}\right] + \varepsilon^2[\dots] \tag{13}$$

Example: Duffing Oscillator

Consider a Duffing Oscillator equation with a Gaussian excitation

$$\ddot{X} + 2\beta \dot{X} + \omega_0^2 \left(X + \varepsilon X^3\right) = Y(t) \tag{14}$$

Suppose we want to find the stationary response X(t). Assuming $E\{Y(t)\} = 0$, we find $E\{X\} = 0$. From Equation (12) it follows that

$$E\{X^2(t)\} = E\{X_0^2(t)\} + 2\varepsilon E\{X_0(t) X_1(t_1)\} + \dots \tag{15}$$

For stationary response, instead of (8) and (9), we find

$$X_0(t) = \int_{-\infty}^{t} h(t-\tau) Y(\tau) d\tau = \int_0^{\infty} h(\tau) Y(t-\tau) d\tau, \tag{16}$$

$$X_1(t) = -\omega_0^2 \int_{-\infty}^{t} h(t-\tau) X_0^3(\tau) d\tau = -\omega_0^2 \int_0^{\infty} h(\tau) X_0^3(t-\tau) d\tau, \tag{17}$$

where h(t) is given by Equation (10) and we set $g = x^3$. Now

$$E\{X_0^2(t)\} = \int_0^{\infty}\int_0^{\infty} h(\tau_1) h(\tau_2) R_{YY}(\tau_1 - \tau_2) d\tau_1 d\tau_2, \tag{18}$$

$$E\{X_0(t) X_1(t)\} = -\omega_0^2 \int_0^{\infty} h(\tau) E\{X_0(t) X_0^3(t-\tau)\} d\tau. \tag{19}$$

Using Equation (16), Equation (19) becomes

$$\begin{aligned} E\{X_0(t) X_1(t)\} = -\omega_0^2 \int_0^{\infty} d\tau h(\tau) \int_0^{\infty} d\tau_1 h(\tau_1) \int_0^{\infty} d\tau_2 h(\tau_2) \int_0^{\infty} d\tau_3 h(\tau_3) \int_0^{\infty} d\tau_4 h(\tau_4) \\ E\{Y(t-\tau_1) Y(t-\tau-\tau_2) Y(t-\tau-\tau_3) Y(t-\tau-\tau_4)\} \end{aligned}. \tag{20}$$

But Y(t) is a zero-mean Gaussian process. Thus,

$$\begin{aligned} E\{Y(t_1)Y(t_2)Y(t_3)Y(t_4)\} = R_{YY}(t_1 - t_2)R_{YY}(t_3 - t_4) \\ + R_{YY}(t_1 - t_3)R_{YY}(t_2 - t_4) \\ + R_{YY}(t_1 - t_4)R_{YY}(t_2 - t_3) \end{aligned} \quad . \tag{21}$$

Let in Equation (20)

$$I_4 = E\{Y(t-\tau_1)Y(t-\tau-\tau_2)Y(t-\tau-\tau_3)Y(t-\tau-\tau_4)\}.$$

Using Equation (21) we find

$$\begin{aligned} I_4 = R_{YY}(\tau - \tau_1 + \tau_2)R_{YY}(\tau_3 - \tau_4) \\ + R_{YY}(\tau - \tau_1 + \tau_3)R_{YY}(\tau_2 - \tau_4). \\ + R_{YY}(\tau - \tau_1 + \tau_4)R_{YY}(\tau_2 - \tau_3) \end{aligned} \tag{22}$$

Employing (22) in Equation (20), the result becomes

$$\begin{aligned} E\{X_0(t)X_1(t)\} = -3\omega_0^2 \int_0^\infty d\tau h(\tau) \int_0^\infty \int_0^\infty d\tau_1 d\tau_2 h(\tau_1)h(\tau_2)R_{YY}(\tau - \tau_1 + \tau_2) \\ \cdot \int_0^\infty \int_0^\infty d\tau_3 d\tau_4 h(\tau_3)h(\tau_4)R_{YY}(\tau_3 - \tau_4) \end{aligned} \quad . \tag{23}$$

Recalling that

$$R_{X_0X_0}(\tau) = \int_0^\infty \int_0^\infty h(\tau_1)h(\tau_2)R_{YY}(\tau - \tau_1 + \tau_2)d\tau_1 d\tau_2 \, , \tag{24}$$

Equation (23) may be restated as

$$E\{X_0(t)X_1(t)\} = -3\omega_0^2 R_{X_0X_0}(0)\int_0^\infty h(\tau)R_{X_0X_0}(\tau)d\tau \, . \tag{25}$$

Therefore, from equation (15) we find

$$E\{X^2(t)\} = R_{X_0X_0}(0)\left[1 - 6\varepsilon\omega_0^2 \int_0^\infty h(\tau)R_{X_0X_0}(\tau)d\tau\right] \tag{26}$$

Equation (26) gives the variance of X up to the first order in ε. Other statistics of X could be found in a similar fashion.