ME 529 - Stochastics

Peturbation Techniques

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Perturbation Techniques

Outline

- Non-Linear System Subject to Random Excitations
- Perturbation Expansion
- Series Solution
- Example of Duffing Oscillator

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A nonlinear system subject to a random excitation is given as

$$\ddot{X} + 2\beta\dot{X}(t) + \omega_0^2\left[X(t) + \varepsilon g(X,\dot{X})\right] = Y(t)$$

For small ε, assume a solution in the form of

$$X(t) = X_0(t) + \varepsilon X_1(t) + \varepsilon^2 X_2(t) + \dots$$

Substitute in the equation and order as powers of ε,

ε^0 ⇒ $$\ddot{X}_0 + 2\beta\dot{X}_0 + \omega_0^2 X_0 = Y(t)$$

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ε^1 ⇒ $$\ddot{X}_1 + 2\beta\dot{X}_1 + \omega_0^2 X_1 = -\omega_0^2 g(X_0,\dot{X}_0)$$

ε^2 ⇒ $$\ddot{X}_2 + 2\beta\dot{X}_2 + \omega_0^2 X_2 = -\omega_0^2\left[\frac{\partial g(X_0,\dot{X}_0)}{\partial X_0}X_1 + \frac{\partial g(X_0,\dot{X}_0)}{\partial \dot{X}_0}\dot{X}_1\right]$$

Where we used $$g(X,\dot{X}) = g(X_0,\dot{X}_0) + \varepsilon\left[\frac{\partial g}{\partial X_0}X_1 + \frac{\partial g}{\partial \dot{X}_0}\dot{X}_1\right] + \varepsilon^2[\dots]$$

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The resulting equations are linear and can be solved

$X(0)=\dot{X}(0)=0$ ⇒ $X_o(0)=\dot{X}_o(0)=0$, $X_1(0)=\dot{X}_1(0)=0$

$$X_0(t)=\int_0^t h(t-\tau)Y(\tau)d\tau$$

$$X_1(t)=-\omega^2\int_0^t h(t-\tau)g\left[X_0(\tau),\dot{X}_0(\tau)\right]d\tau$$

Here

$$h(t)=\frac{1}{\Omega_0}e^{-\beta t}\sin\Omega_0 t \qquad \Omega_0^2=\omega_0^2-\beta^2$$

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Response Statistics

$$E\{X(t)\}=E\{X_0(t)\}+\varepsilon E\{X_1(t)\}+\ldots$$

$$E\{X^2(t)\}=E\{X_0^2(t)\}+2\varepsilon E\{X_0(t)X_1(t)\}+\ldots$$

$$R_{XX}(t_1,t_2)=E\{X_0(t_1)X_0(t_2)\}+\varepsilon\left[E\{X_0(t_1)X_1(t_2)\}+E\{X_0(t_2)X_1(t_1)\}\right]+\varepsilon^2[\ldots]$$

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Duffing Oscillator with Gaussian Excitation

$$\ddot{X}+2\beta\dot{X}+\omega_0^2\left(X+\varepsilon X^3\right)=Y(t)$$

Assume $E\{Y(t)\}=0$ ⇒ $E\{X\}=0$

⇒ $E\{X^2(t)\}=E\{X_0^2(t)\}+2\varepsilon E\{X_0(t)X_1(t_1)\}+\ldots$

Stationary Response

$$X_0(t)=\int_{-\infty}^t h(t-\tau)Y(\tau)d\tau=\int_0^\infty h(\tau)Y(t-\tau)d\tau$$

$$X_1(t)=-\omega_0^2\int_{-\infty}^t h(t-\tau)X_0^3(\tau)d\tau=-\omega_0^2\int_0^\infty h(\tau)X_0^3(t-\tau)d\tau$$

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⇒ $$E\{X_0^2(t)\}=\int_0^\infty\int_0^\infty h(\tau_1)h(\tau_2)R_{YY}(\tau_1-\tau_2)d\tau_1 d\tau_2$$

$$E\{X_0(t)X_1(t)\}=-\omega_0^2\int_0^\infty h(\tau)E\{X_0(t)X_0^3(t-\tau)\}d\tau$$

For Gaussian Processes

$$\begin{aligned}E\{Y(t_1)Y(t_2)Y(t_3)Y(t_4)\}&=R_{YY}(t_1-t_2)R_{YY}(t_3-t_4)\\&+R_{YY}(t_1-t_3)R_{YY}(t_2-t_4)\\&+R_{YY}(t_1-t_4)R_{YY}(t_2-t_3)\end{aligned}$$

Duffing Oscillator

$$E\{X_0(t)X_0^3(t-\tau)\} = 3R_{X_oX_o}(\tau)R_{X_oX_o}(0)$$

Hence

$$E\{X_0(t)X_1(t)\} = -3\omega_0^2 R_{X_0X_0}(0)\int_0^\infty h(\tau)R_{X_0X_0}(\tau)d\tau$$

$$E\{X^2(t)\} = R_{X_0X_0}(0)\left[1 - 6\varepsilon\omega_0^2\int_0^\infty h(\tau)R_{X_0X_0}(\tau)d\tau\right]$$

This gives the variance of X up to the 1st order in ε. Other statistics can be found similarly.

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Concluding Remarks

- **Non-Linear System Subject to a Random Excitation**
- **Perturbation Method**
- **Perturbation Solution**
- **Duffing Oscillator**

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Thank you!

Questions?