Second-Order Systems (Stationary Solutions)

Consider a single-degree-of-freedom system with non-linear spring given as

$$\ddot{X} + \beta\dot{X} + g(X) = n(t), \tag{1}$$

with $n(t)$ being a Gaussian white noise with $R_{nn}(\tau) = 2D\delta(\tau)$. Equation (1) may be restated as

$$\left\{ \begin{array}{c} \dfrac{dX}{dt} = \dot{X} \\ \dfrac{d\dot{X}}{dt} = -\beta\dot{X} - g(X) + n(t) \end{array} \right\}. \tag{2}$$

The corresponding Fokker-Planck equation is given by

$$\frac{\partial f}{\partial t} = -\frac{\partial}{\partial x}(\dot{x}f) + \frac{\partial}{\partial \dot{x}}[(\beta\dot{x} + g(x))f] + D\frac{\partial^2 f}{\partial \dot{x}^2}. \tag{3}$$

The stationary density function satisfies

$$-\dot{x}\frac{\partial f}{\partial x} + \frac{\partial}{\partial \dot{x}}[(\beta\dot{x} + g(x))f] + D\frac{\partial^2 f}{\partial \dot{x}^2} = 0, \tag{4}$$

or

$$\left(-\dot{x}\frac{\partial f}{\partial x} + g(x)\frac{\partial f}{\partial \dot{x}}\right) + \frac{\partial}{\partial \dot{x}}\left[\beta\dot{x}f + D\frac{\partial f}{\partial \dot{x}}\right] = 0. \tag{5}$$

We look for special class of solutions for which

$$\dot{x}\frac{\partial f}{\partial x} + g(x)\frac{\partial f}{\partial \dot{x}} = 0, \tag{6}$$

$$\frac{\partial}{\partial \dot{x}}\left(\beta\dot{x}f + D\frac{\partial f}{\partial \dot{x}}\right) = 0. \tag{7}$$

Solution to Equation (6) is given as

$$f = C(x)e^{-\frac{\beta}{2D}\dot{x}^2} \tag{8}$$

Using (8) in (6) we find

$$-\dot{x}\frac{dc}{dx}e^{-\frac{\beta\dot{x}^2}{2D}}+g(x)\left(-\frac{\beta\dot{x}}{D}\right)Ce^{-\frac{\beta}{2D}\dot{x}^2}=0\,, \tag{9}$$

or

$$\frac{1}{C}\frac{dC}{dx}=-\frac{\beta}{D}g(x),$$

which implies

$$C=C_0e^{-\frac{\beta}{D}G(x)},\qquad G(x)=\int_0^x g(\xi)d\xi\,.$$

Hence

$$f=C_0e^{-\frac{\beta}{D}\left(G(x)+\frac{\dot{x}^2}{2}\right)}$$

Generalized Stationary Solutions

Consider a nonlinear system given by

$$\ddot{X}+h(H)\dot{X}+g(x)=n(t) \tag{1}$$

with $n(t)$ being a white noise, and

$$H=\frac{\dot{X}^2}{2}+\int_0^x g(\eta)d\eta\,. \tag{2}$$

Equation (1) may be restated as

$$\left\{\begin{aligned}\frac{dX}{dt}&=\dot{X}\\ \frac{d\dot{X}}{dt}&=-\left(g(X)+h(H)\dot{X}\right)+n(t)\end{aligned}\right\} \tag{3}$$

The corresponding Fokker-Planck equation is given as

$$\frac{\partial f}{\partial t}=-\dot{x}\frac{\partial f}{\partial x}+\frac{\partial}{\partial\dot{x}}\left[\left(g(x)+h(H)\dot{x}\right)f\right]+D\frac{\partial^2 f}{\partial\dot{x}^2}\,. \tag{4}$$

For stationary conditions, we find

$$-\dot{x}\frac{\partial f}{\partial x}+g(x)\frac{\partial f}{\partial \dot{x}}+\frac{\partial}{\partial \dot{x}}\left[h(H)\dot{x}f+D\frac{\partial f}{\partial \dot{x}}\right]=0\,. \tag{5}$$

Now set

$$-\dot{x}\frac{\partial f}{\partial x}+g(x)\frac{\partial f}{\partial \dot{x}}=0\,, \tag{6}$$

$$\frac{\partial}{\partial \dot{x}}\left[h(H)\dot{x}f+D\frac{\partial f}{\partial \dot{x}}\right]=0\,. \tag{7}$$

We assume that $f = f(H)$, then

$$\frac{\partial f}{\partial \dot{x}}=\frac{\partial f}{\partial H}\dot{x}\,,\ \frac{\partial f}{\partial x}=\frac{\partial f}{\partial H}g(x)\,. \tag{8}$$

Equation (6) is now identically satisfied and Equation (7) becomes

$$h(H)\dot{x}f+D\frac{\partial f}{\partial H}\dot{x}=0\,, \tag{9}$$

or

$$h(H)f+D\frac{\partial f}{\partial H}=0\,, \tag{10}$$

$$\frac{df}{f}=-\frac{1}{D}h(H)dH\,, \tag{11}$$

$$f=C_o e^{-\frac{1}{D}\int_0^H h(\xi)d\xi}\,, \tag{12}$$

with

$$C_o=\left[\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} exp\left\{-\frac{1}{D}\int_0^H h(\xi)d\xi\right\}dx d\dot{x}\right].$$

Additional Exact Steady Solutions

Consider a nonlinear system given as

$$\ddot{X}_i + \beta_i h(H)\dot{X}_i + \frac{\partial V(\mathbf{X})}{\partial X_i} = n_i(t),$$

with

$$R_{n_i n_j}(\tau) = 2D_i \delta_{ij}\delta(\tau), \qquad H = \frac{1}{2}\sum_i \dot{X}_i^2 + V(\mathbf{X}), \qquad \frac{D_i}{\beta_i} = const.$$

Solution:

$$f = C_0 \exp\left\{-\left(\frac{\beta_i}{D_i}\right)\int_0^H f(\xi)d\xi\right\}.$$

Consider

$$\ddot{X} + \left(X^2 + 2\dot{X}^2 - \frac{2}{X^2 + 2\dot{X}^2}\right)2D\dot{X} + \frac{2X^3 + X\dot{X}^2}{X^2 + 2\dot{X}^2} = n(t),$$

with

$$R_{nn}(\tau) = 2D\delta(\tau).$$

Solution:

$$f = A\exp\left\{-\left(x^4 + \dot{x}^4 + x^2\dot{x}^2\right)\right\}\left(x^2 + 2\dot{x}^2\right)$$

Caughey & Ma, (Int. Y. Nonlinear Mech. 17, 137 (1982)) more generally considered

$$\ddot{X} + \left[H_{\dot{x}}h(H) - \frac{H_{\dot{x}\dot{x}}}{H_{\dot{x}}}\right]D\dot{X} + \frac{H_x}{H_{\dot{x}}} = n(t), \qquad H_x = \frac{\partial H}{\partial x}, \quad H_{\dot{x}} = \frac{\partial H}{\partial \dot{x}}.$$

The corresponding Fokker-Planck equation is given as

$$\frac{\partial f}{\partial t} = -\dot{x}\frac{\partial f}{\partial x} + \frac{\partial}{\partial \dot{x}}\left\{\left[\left(H_{\dot{x}}h(H) - \frac{H_{\dot{x}\dot{x}}}{H_{\dot{x}}}\right)D\dot{x} + \frac{H_x}{H_{\dot{x}}}\right]f\right\} + D\frac{\partial^2 f}{\partial \dot{x}^2},$$

with $H(x,\dot{x})>0$, $H_{\dot{x}}>0$.

For steady solution

$$\left\{\begin{array}{c} -\dot{x}\dfrac{\partial f}{\partial x}+\dfrac{\partial}{\partial \dot{x}}\left(\dfrac{H_x}{H_{\dot{x}}}f\right)=0 \\ \dfrac{\partial}{\partial \dot{x}}\left[\left(H_{\dot{x}}h(H)-\dfrac{H_{\dot{x}\dot{x}}}{H_{\dot{x}}}\right)\dot{x}f+\dfrac{\partial f}{\partial \dot{x}}\right]=0 \end{array}\right\}.$$

The exact solution is given by

$$f=C_0\exp\left\{-\int_0^H h(\xi)d\xi\right\}H_{\dot{x}}.$$

Consider

$$\ddot{X}+\beta\,\mathrm{sgn}\,\dot{X}+\left(1+\frac{\beta}{D}\left|\dot{X}\right|\right)g(X)=n(t)$$

Then

$$f=C_0\exp\left\{-\frac{\beta}{D}\left|\dot{X}\right|-\left(\frac{\beta}{D}\right)^{2x}\int_0^x g(\xi)d\xi\right\}.$$

Exact Solutions (Yong & Lin, J. Appl. Mech. June (1987) 414-418)

Consider

$$\ddot{X}+\left[h(\Gamma)+n_1(t)\right]\dot{X}+\omega_o^2\left[1+n_2(t)\right]X=n_3(t)$$

where

$$\Gamma=\frac{1}{2}\dot{X}^2+\frac{1}{2}\omega_o^2X^2$$

The corresponding Fokker-Planck Equation is given as

$$.-\dot{X}\frac{\partial f}{\partial x}+\frac{\partial}{\partial \dot{X}}\left\{\left[h(\Gamma)\dot{X}-D_{22}\dot{X}+\omega_o^2X\right]f\right\}+\frac{\partial^2}{\partial \dot{X}^2}\left[\left(\omega_o^4D_{11}X^2+D_{22}\dot{X}^2+D_{33}\right)f\right]=0$$

Note that the corresponding Ito equation (with Wong-Zakai correction) is given as

$$dX = \dot{X}dt,$$
$$d\dot{X} = -\{(h(\Gamma) - D_{22})\dot{X} + \omega_0^2 X\}dt + \sqrt{2(\omega_0^4 D_{11} X^2 + D_{22}\dot{X}^2 + D_{33})}d\hat{W}.$$

For

$$E\{(d\hat{W})^2\} = dt,$$

the exact solution is

$$f(x,\dot{x}) = \frac{C_3}{\sqrt{\frac{(2D_{22}\Gamma + D_{33})}{\pi}}} exp\left\{-\int_0^\Gamma \frac{h(u)du}{2D_{22}u + D_{33}}\right\}.$$

For $h(\Gamma) = \beta\Gamma + \alpha$, then

$$f(x,\dot{x}) = C\left(\frac{2D_{22}\Gamma + D_{33}}{\pi}\right)^{\frac{1}{2}\left(\beta\frac{D_{33}}{2D_{22}^2} - \frac{\alpha}{D_{22}} - 1\right)} exp\left\{-\frac{\beta\Gamma}{2D_{22}}\right\}.$$

or

$$f(x,\dot{x}) = C_o\, exp\left\{-\frac{\beta\Gamma}{2D_{22}}\right\}.$$

For

$$\ddot{X} + (\alpha + \beta X^2)\dot{X} + \omega_o^2[1 + n_1(t)]X = n_2(t) \qquad \text{with} \qquad \frac{D_{22}}{D_{11}} = \frac{\alpha}{\beta}$$

the solution is

$$f = C_4\, exp\left\{-\frac{\beta}{2D_{11}}(\dot{X}^2 + \omega_o^2 X^2)\right\}, \qquad \text{(Gaussian).}$$

For

$$\ddot{X}_i + h(\Gamma)\dot{X}_i + \omega_i^2 X_i + \sum_{j=1}^{n} n_{ij}(t)\dot{X}_j + \sum_{j=1}^{n} \eta_{ij}(t) X_j = \xi_j(t),$$

with

$$\Gamma = \frac{1}{2}\sum_{j=1}^{n}\left(\dot{X}_j^2 + \omega_j^2 X_j^2\right)$$

and n_{ij}, η_{ij}, and ξ_j are independent white noise processes with

$$\langle n_{ij}(t) n_{ij}(t+\tau)\rangle = 2D_{11}\delta(\tau),$$

$$\langle \eta_{ij}(t) \eta_{ij}(t+\tau)\rangle = 2D_{22}\delta(\tau),$$

$$\langle \xi_j(t) \xi_j(t+\tau)\rangle = 2D_{33}\delta(\tau).$$

The exact solution is

$$f = C_5 (2D_{22}\Gamma + D_{33})^{-\frac{1}{2}} \exp\left[-\int_0^{\Gamma} \frac{h(U)dU}{2D_{22}U + D_{33}}\right].$$