ME 529 - Stochastics

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Fokker-Planck Equation

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Fokker-Planck Equation

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Outline

- Ito's Equation
- Fokker-Planck Equation for Ito's Equation
- General Moment Equation
- Example-1st Order Non-Linear

Fokker-Planck Equation

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Ito's Equation

$$\frac{d\mathbf{X}}{dt} = \mathbf{g}(\mathbf{x}, t) + \mathbf{G} \cdot \mathbf{n}$$

$$d\mathbf{X} = \mathbf{g}(\mathbf{x}, t)dt + \mathbf{G}(\mathbf{x}, t) \cdot d\mathbf{W}$$

with

$$E\{n_i(t+\tau)n_j(t)\} = 2D_{ij}\delta(\tau)$$

$$E\{dW_i dW_j\} = 2D_{ij}dt$$

Theorem

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The joint density function $f_X(x,t)$, with X(t) being solution to Ito's equation, satisfies the Fokker-Planck (Smoluchowski) equation given as :

$$\frac{\partial f}{\partial t} = -\sum_j \frac{\partial}{\partial x_j}\left(g_j(\mathbf{x}, t) f\right) + \sum_i \sum_j \frac{\partial^2}{\partial x_i \partial x_j}\left[\left(GDG^T\right)_{ij} f\right]$$

Proof

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Starting with $f(\mathbf{x},t) = E\{\delta(\mathbf{x}(t) - \mathbf{x})\}$

Taking time derivative and keeping terms up to first order in *dt* (second order in *dW*),

$$\frac{\partial f}{\partial t}dt = \frac{\partial}{\partial t}E\{\delta(\mathbf{X}-\mathbf{x})\}dt$$

$$= E\left\{\sum_j \frac{\partial}{\partial X_j}[\delta(\mathbf{X}-\mathbf{x})]dX_j + \frac{1}{2}\sum_i\sum_j \frac{\partial^2[\delta(\mathbf{X}-\mathbf{x})]}{\partial X_i \partial X_j}dX_i dX_j\right\}$$

Using $dX_j = g_j(\mathbf{x},t)dt + \sum_k G_{jk}dW_k$

Proof

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$$\frac{\partial f}{\partial t}dt = E\left\{-\sum_j \frac{\partial}{\partial x_j}\left[\delta(\mathbf{X}-\mathbf{x})\left(g_j(\mathbf{x},t)dt + \sum_k G_{jk}dW_k\right)\right]\right.$$

$$\left. + \frac{1}{2}\sum_i\sum_j \frac{\partial^2}{\partial x_1 \partial x_j}\left[\delta(\mathbf{X}-\mathbf{x})\left(g_i(\mathbf{x},t)dt + \sum_k G_{ik}dW_k\right)\left(g_j dt + \sum_\ell G_{j\ell}dW_\ell\right)\right]\right\}$$

Noting that dW_k is independent of X(t), and $E\{dW_i dW_j\} = 2D_{ij}dt$ **it follows that**

$$\frac{\partial f}{\partial t} = -\sum \frac{\partial}{\partial x_j}\left(g_j(\mathbf{x},t)f\right) + \sum_i\sum_j \frac{\partial^2}{\partial x_i \partial x_j}\left[f\left(\sum_k\sum_\ell G_{ik}G_{j\ell}D_{k\ell}\right)\right]$$

Moments of Fokker-Planck Equation

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Given Ito's Equation

$$\frac{dX_i}{dt} = g_i(\mathbf{x},t) + \sum_j G_{ij}(\mathbf{x},t)\frac{dW_i}{dt} \qquad n_i(t) = \frac{dW_i}{dt}$$

$$E\{n_i(t)\} = 0 \qquad E\{n_i(t_1)n_j(t_2)\} = 2D_{ij}\delta(t_1 - t_2)$$

Fokker-Planck Equation

$$\frac{\partial f}{\partial t} = -\sum_i \frac{\partial}{\partial x_i}[g_i(\mathbf{x},t)f] + \sum_i\sum_j \frac{\partial^2}{\partial x_i \partial x_j}\left[(GDG^T)_{ij}\cdot f\right]$$

Moments of Fokker-Planck Equation

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Expected value of *h(X)* is given as

$$E\{h(\mathbf{X})\} = \int_{-\infty}^{+\infty}\cdots\int_{-\infty}^{+\infty} h(x)f(\mathbf{x},t\,|\,\mathbf{x}_0,t_0)f(\mathbf{x}_0,t_0)d\mathbf{x}d\mathbf{x}_0$$

Taking Time Rate of Change

$$\frac{d}{dt}E\{h(\mathbf{x})\} = \int_{-\infty}^{+\infty}\cdots\int_{-\infty}^{+\infty} h(\mathbf{x})\frac{\partial f}{\partial t}f(\mathbf{x}_0,t_0)d\mathbf{x}dx_0$$

Moments of Fokker-Planck Equation

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Eliminating ∂f/∂t,

$$\frac{d}{dt}E\{h(\mathbf{x})\}=\int_{-\infty}^{+\infty}\cdots\int_{-\infty}^{+\infty}\left[\begin{array}{c}-\sum_i h(\mathbf{x})\frac{\partial}{\partial x_i}(g_i f)\\ +\sum_i\sum_j h(\mathbf{x})\frac{\partial^2}{\partial x_i\partial x_j}\left((GDG^T)_{ij} f\right)\end{array}\right]f(\mathbf{x}_0,t_0)d\mathbf{x}d\mathbf{x}_0$$

Integrating by parts leads to the general moment equation:

$$\frac{d}{dt}E\{h(\mathbf{x})\}=\sum_i E\left\{\frac{\partial h}{\partial X_i}g_i(\mathbf{x},t)\right\}+\sum_i\sum_j E\left\{(GDG^T)_{ij}\frac{\partial^2 h}{\partial X_i\partial X_j}\right\}$$

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Example-1st Order NonLinear Equation

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Find moment equation corresponding to

$$\dot{X}=-\left[X(t)+aX^3(t)\right]+n(t)\qquad R_{nn}(\tau)=2D\delta(\tau)$$

$$\Rightarrow\quad g=-(X+aX^3)\qquad G=1$$

1st Order System $\dot{X}=g(X,t)+G(X,t)n(t)$

$$\Rightarrow\quad \frac{d}{dt}E\{h(X)\}=E\left\{\frac{\partial h}{\partial x}g\right\}+DE\left\{G^2\frac{\partial^2 h}{\partial X^2}\right\}$$

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Example-1st Order NonLinear Equation

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For $h(X)=X^k \Rightarrow \frac{\partial h}{\partial x}=kx^{k-1}$, $\frac{\partial^2 h}{\partial x^2}=k(k-1)x^{k-2}$

General Moment Equation $m_k=E\{X^k\}$

$$\dot{m}_k=kE\{X^{k-1}g\}+DE\{G^2X^{k-2}\}k(k-1)$$

Or

$$\dot{m}_k=-k(m_k+am_{k+2})+Dk(k-1)m_{k-2}$$

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Example-1st Order NonLinear Equation

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k = 1 $\dot{m}_1=-(m_1(t)+am_3(t))$

k = 2 $\dot{m}_2=-2(m_2(t)+am_4(t))+2D$

k = 3 $\dot{m}_3=-3(m_3(t)+am_5(t))+6Dm_1$

A closure assumption is now needed, e.g.,

$$\begin{cases}m_3=a_0+a_1m_1+a_2m_2\\ m_4=b_0+b_1m_1+b_2m_2\end{cases}$$

Coefficients may be estimated by minimizing mean-square error

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Example-1st Order NonLinear Equation

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$$\begin{Bmatrix} \overline{e_1^2} = E\left\{\left(X^3 - a_0 - a_1 X - a_2 X^2\right)^2\right\} \\ \overline{e_2^2} = E\left\{\left(X^4 - b_0 - b_1 X - b_2 X^2\right)^2\right\} \end{Bmatrix}$$

$$\frac{\partial \overline{e_1^2}}{\partial a_i} = 0 \qquad \frac{\partial \overline{e_2^2}}{\partial b_i} = 0 \qquad i = 0,1,2$$

Alternative closure scheme assumes X(t) is quasi-Gaussian

$$\begin{Bmatrix} \mu_3 = 0 \\ \mu_4 = 3\mu_2^2 \end{Bmatrix}$$

$$\begin{Bmatrix} \mu_3 = E\left\{(X - m_1)^3\right\} = m_3 - m_1 m_2 - 2m_1^2 \\ \mu_4 = E\left\{(X - m_1)^4\right\} = m_4 - 4m_1 m_3 + 6m_1^2 m_2 - 3m_1^4 \end{Bmatrix}$$

Mexico City Earthquake

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Mexico City Earthquake

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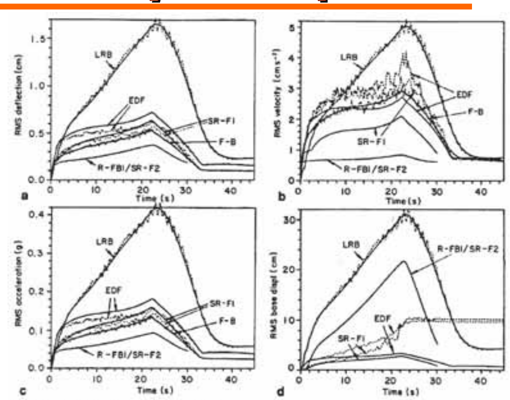

Figure 7 RMS responses for Mexico City 1985 earthquake

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Concluding Remarks

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- Fokker-Planck Equation for Ito's Equation
- General Moment Equation
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