Fokker-Planck Equation

Given Ito's equation

$$\frac{d\mathbf{X}}{dt} = \mathbf{g}(\mathbf{x},t) + \mathbf{G}\cdot\mathbf{n}, \tag{1}$$

or

$$d\mathbf{X} = \mathbf{g}(\mathbf{x},t)dt + \mathbf{G}(\mathbf{x},t)\cdot d\mathbf{W}, \tag{2}$$

with

$$E\{n_i(t+\tau)n_j(t)\} = 2D_{ij}\delta(\tau), \tag{3}$$

or

$$E\{dW_i dW_j\} = 2D_{ij}dt. \tag{4}$$

We now prove the following important theorem.

Theorem

The joint density function $f_{\mathbf{X}}(\mathbf{x},t)$, with $X(t)$ being solution to Equations (1) or (2), satisfies the Fokker-Planck (Smoluchowski) equation given as,

$$\frac{\partial f}{\partial t} = -\sum_j \frac{\partial}{\partial x_j}\left(g_j(\mathbf{x},t)f\right) + \sum_i\sum_j \frac{\partial^2}{\partial x_i \partial x_j}\left[\left(GDG^T\right)_{ij} f\right]. \tag{5}$$

Proof: Recall

$$f(\mathbf{x},t) = E\{\delta(\mathbf{x}(t)-\mathbf{x})\},$$

$$\frac{\partial f}{\partial t}dt = \frac{\partial}{\partial t}E\{\delta(\mathbf{X}-\mathbf{x})\}dt$$
$$= E\left\{\sum_j \frac{\partial}{\partial X_j}[\delta(\mathbf{X}-\mathbf{x})]dX_j + \frac{1}{2}\sum_i\sum_j \frac{\partial^2[\delta(\mathbf{X}-\mathbf{x})]}{\partial X_i \partial X_j}dX_i dX_j\right\},$$

or

$$\frac{\partial f}{\partial t}dt = E\left\{-\sum_j \frac{\partial}{\partial x_j}\left[\delta(\mathbf{X}-\mathbf{x})\left(g_j(\mathbf{x},t)dt + \sum_k G_{jk}dW_k\right)\right]\right.$$
$$\left. + \frac{1}{2}\sum_i\sum_j \frac{\partial^2}{\partial x_1 \partial x_j}\left[\delta(\mathbf{X}-\mathbf{x})\left(g_i(\mathbf{x},t)dt + \sum_k G_{ik}dW_k\right)\left(g_j dt + \sum_\ell G_{j\ell}dW_\ell\right)\right]\right\}.$$

Noting that dW_k is independent of $\mathbf{X}(t)$, and using (4) we find

$$\frac{\partial f}{\partial t} = -\sum \frac{\partial}{\partial x_j}\left(g_j(\mathbf{x},t) f\right) + \sum_i\sum_j \frac{\partial^2}{\partial x_i \partial x_j}\left[f\left(\sum_k\sum_\ell G_{ik}G_{j\ell}D_{k\ell}\right)\right].$$

Moments of Fokker-Planck Equation

Given Ito's equation

$$\frac{dX_i}{dt} = g_i(\mathbf{x},t) + \sum_j G_{ij}(\mathbf{x},t)\frac{dW_i}{dt} \tag{1}$$

with $n_i(t) = \frac{dW_i}{dt}$ being white noise processes with the statistics

$$E\{n_i(t)\} = 0, \quad E\{n_i(t_1)n_j(t_2)\} = 2D_{ij}\delta(t_1 - t_2) \tag{2}$$

The corresponding Fokker-Planck equation is given by

$$\frac{\partial f}{\partial t} = -\sum_i \frac{\partial}{\partial x_i}[g_i(\mathbf{x},t) f] + \sum_i\sum_j \frac{\partial^2}{\partial x_i \partial x_j}\left[\left(GDG^T\right)_{ij}\cdot f\right]. \tag{3}$$

The expected value of an arbitrary function $h(\mathbf{X})$ is given by

$$E\{h(\mathbf{X})\} = \int_{-\infty}^{+\infty}\dots\int_{-\infty}^{+\infty} h(x) f(\mathbf{x},t \mid \mathbf{x}_0, t_0) f(\mathbf{x}_0, t_0) d\mathbf{x} d\mathbf{x}_0 \,. \tag{4}$$

Time rate of change of (4) is given as

$$\frac{d}{dt}E\{h(\mathbf{x})\} = \int_{-\infty}^{+\infty}\dots\int_{-\infty}^{+\infty} h(\mathbf{x}) \frac{\partial f}{\partial t} f(\mathbf{x}_0, t_0) d\mathbf{x} dx_0 \,. \tag{5}$$

Eliminating $\frac{\partial f}{\partial t}$ between Equations (4) and (5), we find

$$\frac{d}{dt}E\{h(\mathbf{x})\} = \int_{-\infty}^{+\infty}\cdots\int_{-\infty}^{+\infty}\left[\begin{array}{c} -\sum_i h(\mathbf{x})\frac{\partial}{\partial x_i}(g_i f) \\ +\sum_i\sum_j h(\mathbf{x})\frac{\partial^2}{\partial x_i \partial x_j}\left(\left(GDG^T\right)_{ij} f\right)\end{array}\right] f(\mathbf{x}_0, t_0)\, d\mathbf{x}\, d\mathbf{x}_0 \,. \tag{6}$$

Integrating by part, we obtain

$$\frac{d}{dt}E\{h(\mathbf{x})\} = \sum_i E\left\{\frac{\partial h}{\partial X_i} g_i(\mathbf{x},t)\right\} + \sum_i\sum_j E\left\{\left(GDG^T\right)_{ij}\frac{\partial^2 h}{\partial X_i \partial X_j}\right\}. \tag{7}$$

Equation (7) is a general moment equation.

Example

Find the moment equation corresponding to the first order non-linear system given by

$$\dot{X} = -\left[X(t) + aX^3(t)\right] + n(t), \; R_{nn}(\tau) = 2D\delta(\tau). \tag{8}$$

In this case

$$g = -\left(X + aX^3\right), \; G = 1.$$

For a general first order system

$$\dot{X} = g(X,t) + G(X,t)n(t), \tag{9}$$

the moment equation (7) reduces to

$$\frac{d}{dt}E\{h(X)\} = E\left\{\frac{\partial h}{\partial x} g\right\} + DE\left\{G^2\frac{\partial^2 h}{\partial X^2}\right\}. \tag{10}$$

For $h(X) = X^k$, we find, $\frac{\partial h}{\partial x} = kx^{k-1}$, $\frac{\partial^2 h}{\partial x^2} = k(k-1)x^{k-2}$ and.

$$\dot{m}_k = kE\left\{X^{k-1}g\right\} + DE\left\{G^2 X^{k-2}\right\}k(k-1), \; m_k = E\left\{X^k\right\}. \tag{11}$$

For Equation (8), the result is

$$\dot{m}_k = -k\left(m_k + am_{k+2}\right) + Dk(k-1)m_{k-2}\,. \tag{12}$$

For $k = 1,2,\ldots$, equation (12) generates the hierarchy of the moment equations. These are

$$k = 1 \quad \dot{m}_1 = -\left(m_1(t) + a m_3(t)\right), \tag{13}$$

$$k = 2 \quad \dot{m}_2 = -2\left(m_2(t) + a m_4(t)\right) + 2D, \tag{14}$$

$$k = 3 \quad \dot{m}_3 = -3\left(m_3(t) + a m_5(t)\right) + 6Dm_1. \tag{15}$$

Clearly equations (13) through (15) are coupled and a closure assumption is needed.

One possible assumption is

$$\begin{Bmatrix} m_3 = a_0 + a_1 m_1 + a_2 m_2 \\ m_4 = b_0 + b_1 m_1 + b_2 m_2 \end{Bmatrix}, \tag{16}$$

where a's and b's are constant coefficients. These coefficients may be estimated by minimizing the following mean-square errors:

$$\begin{Bmatrix} \overline{e_1^2} = E\left\{\left(X^3 - a_0 - a_1 X - a_2 X^2\right)^2\right\} \\ \overline{e_2^2} = E\left\{\left(X^4 - b_0 - b_1 X - b_2 X^2\right)^2\right\} \end{Bmatrix}. \tag{17}$$

That is, by setting

$$\frac{\partial \overline{e_1^2}}{\partial a_i} = 0, \quad \frac{\partial \overline{e_2^2}}{\partial b_i} = 0, \quad i = 0,1,2. \tag{18}$$

Alternative closure scheme is to assume $X(t)$ is quasi-Gaussian. i.e.,

$$\begin{Bmatrix} \mu_3 = 0 \\ \mu_4 = 3\mu_2^2 \end{Bmatrix}, \tag{19}$$

where μ's are central moments.

$$\begin{Bmatrix} \mu_3 = E\left\{\left(X - m_1\right)^3\right\} = m_3 - m_1 m_2 - 2m_1^2 \\ \mu_4 = E\left\{\left(X - m_1\right)^4\right\} = m_4 - 4m_1 m_3 + 6m_1^2 m_2 - 3m_1^4 \end{Bmatrix}. \tag{20}$$

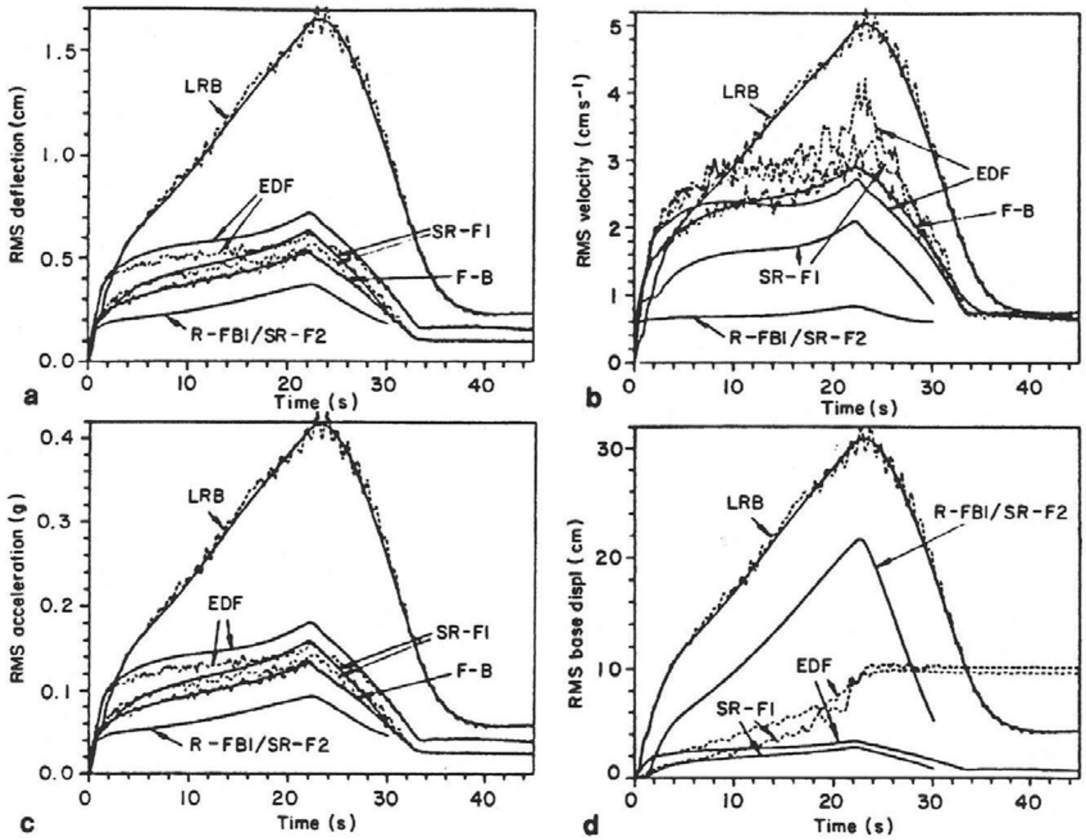

Figure 7 RMS responses for Mexico City 1985 earthquake

Gaussian statistics of the response used in the linearization scheme.

From *Figure 7* it is observed that the LRB system significantly amplifies the RMS response of the structure for this long period earthquake due to resonance. The EDF and the SR-F1 base isolation systems also do not reduce the RMS deflection, velocity and acceleration responses of the structure. The R-FBI and the SR-F2 base isolators are the only ones which appear to reduce the deflection, velocity and acceleration RMS response to a certain extent.

Figure 7 shows that the peak RMS displacement response for the LRB system reaches about 30 cm due to the resonance between the earthquake excitation and the base isolator. The corresponding RMS base displacement for the R-FB1/SR-F2 systems is about 22 cm. The linearization methods predicts RMS base displacements of about 3 to 4 cm for the EDF and the SR-F1 base isolation systems. The Monte-Carlo simulations, however, lead to a relatively large residual base displacement of about 10 cm. The discrepancy is due to the approximations involved in the Gaussian linearization method and in estimating the mean stick-slip parameter[22,36]. It is observed from *Figure 6*, that large base displacement occurs at $t \simeq 24$ (about the time of peak excitation) which essentially remains as a residual base displacement. This peculiar behaviour is not properly accounted for in the present linearization method. From *Figure 7*, it is also observed that the RMS responses reach their peak values at about 20 to 22 s and then decay rapidly and approach their stationary limits.

Response spectra

The 3σ-estimates are used in this section for evaluating peak structural responses. The statistically evalued response spectra for various base-isolated structures are shown in *Figure 8* by the solid lines. The response spectra for the fixed-base structure are also reproduced in this figure for comparison. The dashed lines in *Figure 8* correspond to the peak responses obtained for the accelerogram of the N90W component of Mexico City 1985 earthquake.

Figure 8 shows that as T_1 increases the peak structural deflections and velocities increase, while the peak accelerations remain almost constant. The three standard deviation estimates provide realistic upper bounds on the peak responses for the base-isolated structure under the Mexico City 1985 earthquake excitation. It is also observed from ths figure that peak responses for the base-isolated structure with the LRB, the EDF and the SR-F1 systems are higher than those of the fixed-base

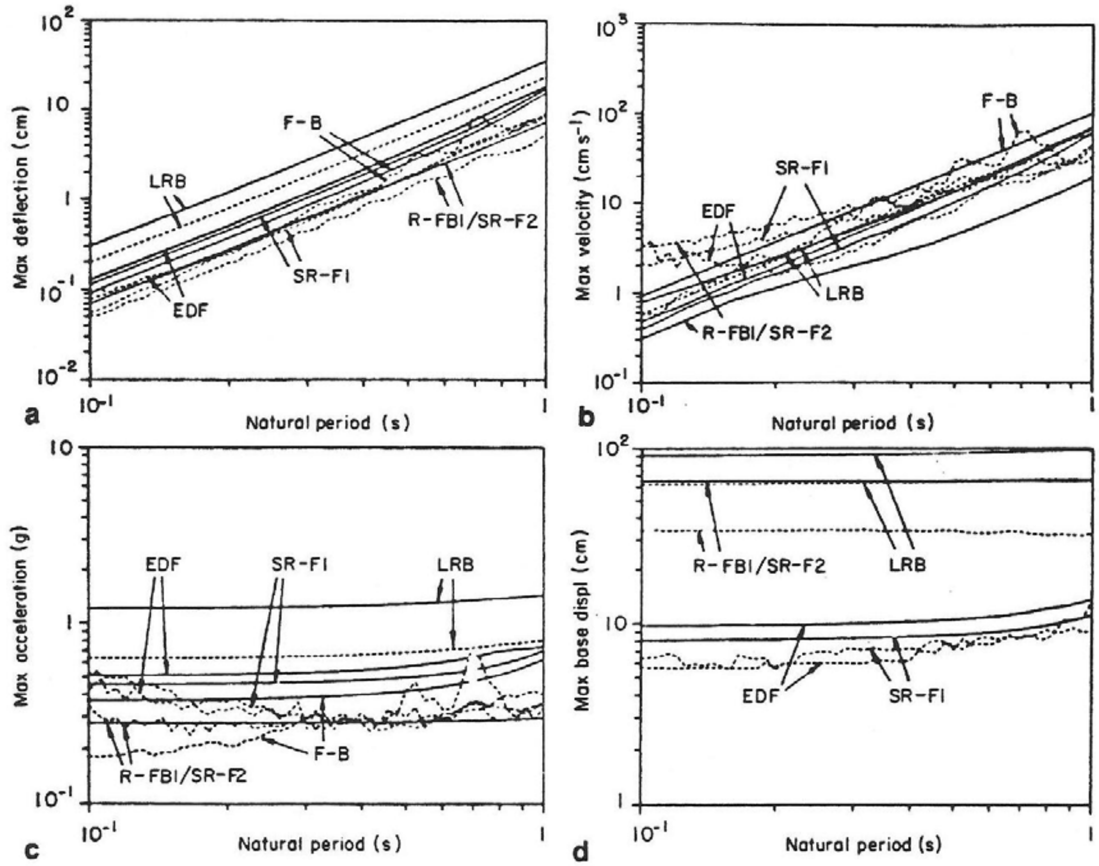

Figure 8 Variations of peak responses of structure with its natural period for Mexico City 1985 earthquake

structure by a factor of 1.2 to 2.5. Thus, the base isolation systems appear not to function properly for this long-period earthquake.

Figure 8(d) compares the peak base displacements of various base-isolated structures. The statistically estimated peak base raft displacements appear to be in reasonable agreement with those obtained for the actual Mexico City 1985 earthquake accelerogram. This figure also shows that the peak displacements are almost constant for the entire range of the natural period considered. It is also observed that the SR-F1 and the EDF systems generate the lowest maximum base displacements among the isolators considered while the LRB system produces the largest one.

The presented results show that the base isolated structures are quite sensitive to long period ground excitations. Therefore, the use of base isolation systems with the typical values of parameters as listed in *Table 2* in regions which have the potential of generating earthquakes with considerable energy at low frequencies should be avoided. It is, of course, conceivable that new base isolation systems may be developed, or the existing ones may be redesigned in order to reduce the sensitivity to long period ground excitations.

Conclusions

Earthquake responses of a base-isolated shear beam structure with different base isolation systems have been probabilistically analysed. In contrast to the earlier studies which used simple white or filtered white noise ground motion models, here, the recently developed models with evolving amplitude and frequency for the El Centro 1940 and the Mexico City 1985 earthquakes are used as seismic excitations. The method of time-dependent equivalent linearization is utilized and the mean-square response statistics of the base-isolated shear beam structure are evaluated. Statistically estimated peak responses of the base-isolated structure are compared with the response spectra for actual earthquake accelerograms.

Based on the results presented, the following conclusions may be drawn

- The nonstationary (quasi-Gaussian) equivalent linearization technique combined with the second-order moment equations provides a systematic and computationally efficient tool for response analysis of base-isolated structures to random earthquake