ME 529 - Stochastics

Markov Processes

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Markov Processes

Outline

Markov Processes

A stochastic process X(t) is called a Markov process if for every n and any $t_1 < t_2 < \ldots < t_n$, its conditional probability satisfies the following condition

$$P\{X(t_n) \le x_n \mid X(t_{n-1}), X(t_{n-2}), \ldots, X(t_1)\} = P\{X(t_n) \le x_n \mid X(t_{n-1})\}$$

or

$$P\{X(t_n) \le x_n \mid X(t) \ for\ all\ t \le t_{n-1}\} = P\{X(t_n) \le x_n \mid X(t_{n-1})\}$$

Important Properties

i) X(t) is also Markov in reverse:

$$P\{X(t_1) \le x_1 \mid X(t) \ for\ all\ t \ge t_2\} = P\{X(t_1) \le x_1 \mid X(t_2)\}$$

ii) The future is independent of the past under the given condition of the present.

iii) If for any $t_1 < t_2$, $X(t_2) - X(t_1)$ is independent of X(t) for $t \le t_1$, the X(t) is a Markov process. Thus, independent increment processes (Poisson, Wiener-Levy) are Markov processes.

Important Properties

iv) $E\{X(t_n)\,|\,X(t_{n-1}),\ldots,X(t_1)\} = E\{X(t_n)\,|\,X(t_{n-1})\}$

v) X(t) is associated with a 1st order equation,

$$\frac{dX}{dt} - \beta(X,t) = j(t) \qquad j(t) = \frac{dW}{dt}$$

with solution $X(t) = X(t_0) + \int_{t_0}^{t} \beta(X(\tau),\tau)\,d\tau + \int_{t_0}^{t} j(\tau)\,d\tau$

vi) Conditional probability density satisfies

$$f_X(x_n,t_n\,|\,x_{n-1},t_{n-1};\ldots;x_2,t_2;x_1,t_1) = f_X(x_n,t_n\,|\,x_{n-1},t_{n-1})$$

Important Properties

vi) The following relation holds

$$f_X(x_1,t_1;x_2,t_2;x_3,t_3) = f_X(x_3,t_3\,|\,x_2,t_2)\,f_X(x_2,t_2\,|\,x_1,t_1)\,f_X(x_1,t_1)$$

$$f_X(x_1,t_1;\ldots;x_n,t_n) = f_X(x_n,t_n\,|\,x_{n-1},t_{n-1})\ldots f_X(x_2,t_2\,|\,x_1,t_1)\,f_X(x_1,t_1)$$

If $X(0) = x_o$ **then** $f_X(x_1,t_1) = f_X(x_1,t_1\,|\,x_o,0)$

A Markov Process is fully specified if:

- **a) Given 1st order density and transition probability density;**
- **b) 2nd order density;**
- **c) Transition density and X(0)**

Chapman-Kolmogorov-Smoluchowski Equation

For Continuous Random Process

$$f(x,t\,|\,x_0,t_0) = \int_{-\infty}^{+\infty} f(x,t;x_1,t_1\,|\,x_0,t_0)\,dx_1 = \int_{-\infty}^{+\infty} f(x,t\,|\,x_1,t_1;x_0,t_0)\,f(x_1,t_1\,|\,x_0,t_0)\,dx_1$$

For Markov Processes

$$f(x,t\,|\,x_1,t_1;x_0,t_0) = f(x,t\,|\,x_1,t_1)$$

$$f(x,t\,|\,x_0,t_0) = \int_{-\infty}^{+\infty} f(x,t\,|\,x_1,t_1)\,f(x_1,t_1\,|\,x_0,t_0)\,dx_1$$

Fokker-Planck Equation

$$\frac{\partial f}{\partial t} = -\frac{\partial}{\partial x}\left[\alpha_1(x,t)f\right] + \frac{1}{2}\frac{\partial^2}{\partial x^2}\left[\alpha_{11}(x,t)f\right]$$

$$\alpha_1(x,t) = \lim_{dt\to 0}\frac{1}{dt}E\{dX(t)\,|\,X(t)=x\} \qquad \alpha_{11}(x,t) = \lim_{dt\to 0}\frac{1}{dt}E\{[dX(t)]^2\,|\,X(t)=x\}$$

Kolmogorov Equation

$$\frac{\partial f}{\partial t_0} + \alpha_1(x_0,t_0)\frac{\partial f}{\partial x_0} + \frac{\alpha_{11}(x_0,t_0)}{2}\frac{\partial^2 f}{\partial x_0^2} = 0$$

$$f = f(x,t\,|\,x_0,t_0)$$

Example: Wiener-Levy Process

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Recall $E\{W(t)\}=0$ $E\{W^2(t)\}=2Dt$ $R_{WW}(t_1,t_2)=\begin{cases}2Dt_1 & t_2\ge t_1\\ 2Dt_2 & t_1\ge t_2\end{cases}$

$E\{W(t_2)-W(t_1)\}=0$ $E\{[W(t_2)-W(t_1)]^2\}=2D(t_2-t_1)$ $t_2>t_1$

Let $t_2=t+dt$ $t_1=t$ $dW=W(t+dt)-W(t)$

⇒ $E\{dW\}=0$ $E\{(dW)^2\}=2Ddt$

⇒ $\alpha_1=\lim_{dt\to0}\frac{1}{dt}E\{dW\,|\,W\}=0$ $\alpha_{11}=\lim_{dt\to0}\frac{1}{dt}E\{(dW)^2\,|\,W\}=2D$

Example: Wiener-Levy Process

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Corresponding Fokker-Planck Equation

$$\frac{\partial f}{\partial t}=-\frac{\partial}{\partial x}(\alpha_1 f)+\frac{1}{2}\frac{\partial^2}{\partial x^2}(\alpha_{11}f)=D\frac{\partial^2 f}{\partial x^2} \Rightarrow \frac{\partial f}{\partial t}=D\frac{\partial^2 f}{\partial W^2}$$

For $f(w,t_o\,|\,w_o,t_o)=\delta(w-w_o)$ **solution is**

$$f=\frac{e^{-\frac{(w-w_o)^2}{4D(t-t_o)}}}{\sqrt{4\pi D(t-t_o)}}$$

Fokker-Planck Equation For Vector-Markov Process

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$$\frac{\partial f}{\partial t}=-\sum_{j=1}^{n}\frac{\partial}{\partial x_j}\left[\alpha_j(\mathbf{x},t)f\right]+\frac{1}{2}\sum_i\sum_j\frac{\partial^2}{\partial x_i\partial x_j}\left[\alpha_{ij}(\mathbf{x},t)f\right]$$

$$\alpha_j=\lim_{dt\to0}\frac{1}{dt}E\{dX_j(t)\,|\,\mathbf{X}(t)=\mathbf{x}\}$$

$$\alpha_{ij}=\lim_{dt\to0}\frac{1}{dt}E\{dX_i(t)dX_j(t)\,|\,\mathbf{X}(t)=\mathbf{x}\}$$

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Ito's Equation $\frac{d\mathbf{X}}{d\mathbf{t}}=\mathbf{g}(\mathbf{X},t)+\mathbf{G}(\mathbf{x},t)\cdot\mathbf{n}(t)$ **or**

$\frac{dX_i}{dt}=g_i(\mathbf{X},t)+\sum_j G_{ij}(\mathbf{X},t)n_j(t)$ $d\mathbf{X}=\mathbf{g}(\mathbf{X},t)dt+\mathbf{G}(\mathbf{X},t)\cdot d\mathbf{W}$

Here n and W being vector white noise and Wiener processes with

$E\{n_i\}=E\{dW_i\}=0$ $E\{n_i(t+\tau)n_j(t)\}=2D_{ij}\delta(\tau)$

$E\{dW_i dW_j\}=2D_{ij}dt$

Fokker-Planck Equation

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Fokker-Planck Equation for Ito's Equation

$$a_j = \lim_{dt \to 0} \frac{1}{dt} E\{dX_j \mid \mathbf{X} = \mathbf{x}\} = g_j(x,t)$$

$$a_{ij} = \lim_{dt \to 0} \frac{1}{dt} E\{dX_i dX_j \mid \mathbf{X} = \mathbf{x}\}$$

$$dX_i dX_j = g_i g_j (dt)^2 + g_i \sum_k G_{jk} dW_k dt + g_j \sum_k G_{ik} dW_k dt + \sum_k \sum_\ell G_{ik} G_{j\ell} dW_k dW_\ell$$

$$\alpha_{ij} = 2 \sum_k \sum_\ell G_{ik} G_{j\ell} D_{k\ell} = 2(\mathbf{G} \cdot \mathbf{D} \cdot \mathbf{G}^T)_{ij}$$

$$\Rightarrow \quad \frac{\partial f}{\partial t} = -\sum_j \frac{\partial}{\partial x_j}\left[g_j(\mathbf{x},t) f\right] + \sum_i \sum_j \frac{\partial^2}{\partial x_i \partial x_j}\left[(\mathbf{G} \cdot \mathbf{D} \cdot \mathbf{G}^T)_{ij} f\right]$$

Example: 1st Order System

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Consider $\frac{dx}{dt} = g(x,t) + G(x,t)\frac{dW}{dt}$ $\quad E\{(dW)^2\} = 2D\,dt$

Fokker-Planck $\frac{\partial f}{\partial t} = -\frac{\partial}{\partial x}(gf) + D\frac{\partial^2}{\partial x^2}(G^2 f)$

Stationary $\frac{d}{dx}\left[-gf + D\frac{d}{dx}(G^2 f)\right] = 0$ $\quad -gf + D\frac{d}{dx}(G^2 f) = c_1 = 0$

Let $G^2 f = F$ $\quad D\frac{dF}{dx} = \frac{g}{G^2}F$ $\quad \frac{dF}{F} = \frac{g}{DG^2}dx$

$$F = C\exp\left\{+\int_0^x \frac{g(x_1)}{DG^2(x_1)}dx_1\right\} \Rightarrow f = \frac{c}{G^2}\exp\left\{\int_0^x \frac{g(x_1)}{DG^2(x_1)}dx_1\right\}$$

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Concluding Remarks

- Markov Processes
- Important Properties
- Chapman-Kolmogorov-Equation
- Fokker-Planck Equation
- Fokker-Planck Equation for Vector Markov Processes
- Fokker-Planck Equation for Ito's Equation

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Thank you!

Questions?