ME 529 - Stochastics

Response of Single-Degree-of Freedom System to a White Excitation

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Response to White Excitation

Outline

- Single Degree of Freedom System
- Nonstationary & Transient Responses
- Examples
- Double Fourier Transforms
- Fourier Transforms of Stochastic Processes
- Response of Non-stationary System

Response to White Excitation

Single Degree of Freedom

$$\ddot{X}(t) + 2\omega_o \xi \dot{X}(t) + \omega_o^2 X(t) = n(t)$$

Initial Conditions

$$X(0) = \dot{X}(0) = 0$$

***n(t)* - White Noise**

$$E\{n(t)\} = 0$$

$$R_{nn}(\tau) = 2\pi S_o \delta(\tau)$$

Response to White Excitation

Impulse Response Function

$$h(t) = \frac{1}{\omega_d} e^{-\xi\omega_o t} \sin \omega_d t$$

$$\omega_d = \omega_o (1 - \xi^2)^{\frac{1}{2}}$$

System Response

$$X(t) = \int_0^t h(t-\tau) n(\tau) d\tau$$

Response Mean

$$E\{X(t)\} = 0$$

Response to White Excitation

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Mean-Square Response

$$E\{X^2(t)\} = \int_0^t\int_0^t h(t-\tau_1)h(t-\tau_2)R_{nn}(\tau_1-\tau_2)\,d\tau_1 d\tau_2$$

For white excitation, integrating over delta function

$$E\{X^2(t)\} = 2\pi S_o \int_0^t h^2(\tau)\,d\tau$$

$$\sigma_x^2(t) = \frac{2\pi S_o}{\omega_d^2}\int_0^t e^{-2\xi\omega_o\tau}\sin^2\omega_d\tau\,d\tau$$

$$\sigma_x^2(t) = \frac{2\pi S_o}{2\xi\omega_o^3}\left\{1-\frac{1}{\omega_d^2}e^{-2\xi\omega_o t}\left[\omega_d^2+2(\xi\omega_o)^2\sin^2\omega_d t+\omega_o\omega_d\xi\sin 2\omega_d t\right]\right\}$$

Response to Broad Band Excitation

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Mean-Square Response

$$\sigma_x^2(t) = \int_0^t\int_0^t h(t-\tau_1)h(t-\tau_2)\frac{1}{\pi}\int_0^\infty S_{nn}(\omega)\cos\omega(\tau_1-\tau_2)\,d\omega\,d\tau_1 d\tau_2$$

Approximate Solution for Small Damping, Caughey and Stumpf (1963)

$$\sigma_x^2(t) \approx \frac{S_{nn}(\omega_o)}{4\xi\omega_o^3}\left\{1-\frac{1}{\omega_d^2}e^{-2\xi\omega_o t}\left[\omega_d^2+2(\xi\omega_o)^2\sin^2\omega_d t+\omega_o\omega_d\xi\sin 2\omega_d t\right]\right\}$$

For $\xi = 0$

$$\sigma_x^2(t) = \frac{S_{nn}(\omega_o)}{4\omega_o^2}\left[2\omega_o t-\sin 2\omega_o t\right]$$

Earthquake Response

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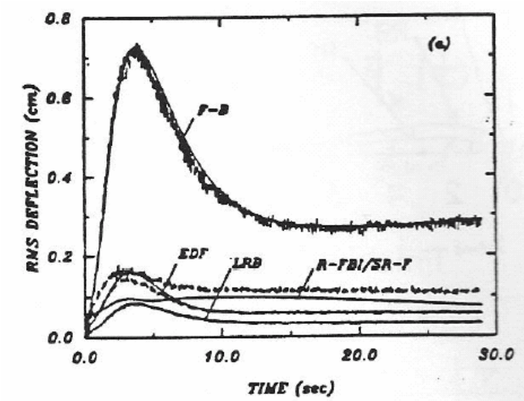

Earthquake Response

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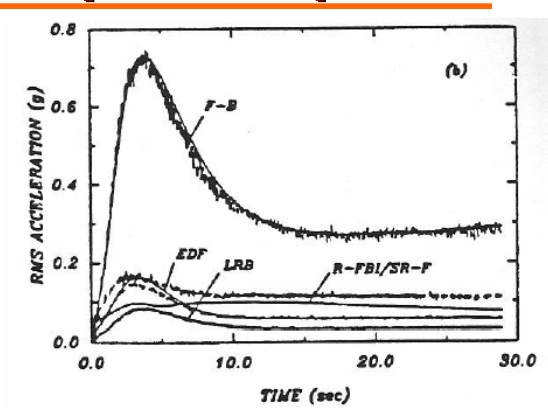

Nonstationary & Transient Response Analysis

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Linear System

$$Y(t)=\int_{-\infty}^{+\infty}X(t-\tau)h(\tau)d\tau$$

Expected Value

$$\eta_Y(t)=\int_{-\infty}^{+\infty}\eta_X(t-\tau)h(\tau)d\tau=\eta_X(t)*h(t)$$

Autocorrelation

$$R_{XY}(t_1,t_2)=\int_{-\infty}^{+\infty}R_{XX}(t_1,t_2-\tau)h(\tau)d\tau=R_{XX}(t_1,t_2)*h(t_2)$$

$$R_{YY}(t_1,t_2)=\int_{-\infty}^{+\infty}R_{XY}(t_1-\tau,t_2)h(\tau)d\tau=R_{XY}(t_1,t_2)*h(t_1)$$

$$\Rightarrow\quad R_{YY}(t_1,t_2)=R_{XX}(t_1,t_2)*h(t_2)*h(t_1)$$

Double Fourier Transforms

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$$\Gamma(\omega_1,\omega_2)=\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}R(t_1,t_2)e^{-i(\omega_1 t_1-\omega_2 t_2)}dt_1dt_2$$

Inverse

$$R(t_1,t_2)=\frac{1}{(2\pi)^2}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\Gamma(\omega_1,\omega_2)e^{i(\omega_1 t_1-\omega_2 t_2)}d\omega_1d\omega_2$$

Properties $\Gamma(\omega_1,\omega_2)=\Gamma^*(\omega_2,\omega_1)$ $\quad\Gamma(-\omega_1,-\omega_2)=\Gamma^*(\omega_1,\omega_2)$

Linear System Response

$$\Gamma_{YY}(\omega_1,\omega_2)=\Gamma_{XX}(\omega_1,\omega_2)H(\omega_1)H^*(\omega_2)$$

X(t) stationary $\Gamma(\omega_1,\omega_2)=2\pi S(\omega_1)\delta(\omega_2-\omega_1)$

Fourier Transforms of Stochastic Processes

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$$\overline{X}(\omega)=\int_{-\infty}^{+\infty}X(\omega)e^{-i\omega t}dt$$ **Exists iff** $$\int_{-\infty}^{+\infty}|X(t)|^2dt<\infty$$

$\therefore$ FT of stationary processes does not exist

Inverse

$$X(t)=\frac{1}{2\pi}\int_{-\infty}^{+\infty}\overline{X}(\omega)e^{i\omega t}d\omega$$

Mean of X(ω)

$$E\{\overline{X}(\omega)\}=\int_{-\infty}^{+\infty}E\{X(t)\}e^{-i\omega t}dt=\int_{-\infty}^{+\infty}\eta_X(t)e^{-i\omega t}dt$$

Theorem

$$E\{X(\omega_1)X^*(\omega_2)\}=\Gamma(\omega_1,\omega_2)$$

Nonstationary Response

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For random input, X(t)

$$Y(t)=h(t)*X(t)\quad\Rightarrow\quad\overline{Y}(\omega)=H(\omega)\overline{X}(\omega)$$

Noting

$$\Gamma_{YY}(\omega_1,\omega_2)=E\{\overline{Y}(\omega_1)\overline{Y}^*(\omega_2)\}$$

$$\Rightarrow\quad\Gamma_{YY}(\omega_1,\omega_2)=H(\omega_1)H^*(\omega_2)\Gamma_{XX}(\omega_1,\omega_2)$$