Linear Systems

Consider a linear system given as

$$LY = X(t), \quad \text{with} \quad Y(0) = 0,$$

where L is a linear differential operator. If h(t) is the impulse response of the system, then

$$Y(t) = \int_0^t h(t-\tau) X(\tau) d\tau = h(t) * X(t) = \int_0^t X(t-\tau) h(\tau) d\tau$$

Here '*' stands for convolution integral. The Autocorrelation of *Y(t)* then is give as

$$R_{YY}(t_1, t_2) = \int_0^{t_1} d\tau_1 \int_0^{t_2} d\tau_2 h(t_1 - \tau_1) h(t_2 - \tau_2) R_{XX}(t_1, t_2),$$

or

$$R_{YY}(t_1, t_2) = h(t_1) * R_{XX}(t_1, t_2) * h(t_2).$$

If $X(t)$ is a white noise with $R_{XX}(\tau) = \alpha\delta(\tau)$, then

$$R_{YY}(t_1, t_2) = \int_0^{t_1} d\tau_1 \int_0^{t_2} d\tau_2 h(t_1 - \tau_1) h(t_2 - \tau_2) \delta(\tau_1 - \tau_2).$$

For $t_1 < t_2$,

$$R_{YY}(t_1, t_2) = \alpha \int_0^{t_1} d\tau_1 h(t_1 - \tau_1) h(t_2 - \tau_1),$$

and

$$E\{Y^2(t)\} = \alpha \int_0^t h^2(t-\tau) d\tau = \alpha \int_0^t h^2(\tau) d\tau.$$

Stationary Excitation and Stationary Response

For stationary excitations, and for stationary response of the system

$$Y(t) = \int_{-\infty}^{+\infty} h(t-\tau) X(\tau) d\tau = h(t) * X(t) = \int_{-\infty}^{+\infty} h(\tau) X(t-\tau) d\tau.$$

But $h(t) = 0$ for $t < 0$; thus,

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$$Y(t) = \int_{-\infty}^{t} h(t-\tau) X(\tau) d\tau = \int_{0}^{\infty} h(\tau) X(t-\tau) d\tau \,.$$

The corresponding autocorrelation becomes

$$R_{YY}(t_1 - t_2) = \int_{-\infty}^{t_1} d\tau_1 \int_{-\infty}^{t_2} d\tau_2 h(t_1 - \tau_1) h(t_2 - \tau_2) R_{XX}(\tau_1 - \tau_2) = \int_{0}^{\infty} d\tau_1 \int_{0}^{\infty} d\tau_2 h(\tau_1) h(\tau_2) R_{XX}(t_1 - t_2 + \tau_2 - \tau_1)$$

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For white excitations, we find

$$R_{YY}(t_1 - t_2) = \alpha \int_{-\infty}^{t_1} d\tau_1 h(t_1 - \tau_1) h(t_2 - \tau_1) \text{ for } t_1 < t_2 \,.$$

And

$$E\{Y^2(t)\} = \alpha \int_{-\infty}^{t} h^2(\tau) d\tau \,.$$