ME 529 - Stochastics

Analysis of Linear Systems

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Analysis of Linear Systems

Outline

- Deterministic Systems
- Random Response of Linear Systems
- Stationary Response Analysis
- Autocorrelation & Cross-Correlation
- System Identification
- Spectral Response
- Examples

Analysis of Linear Systems

Deterministic Systems

***L*: deterministic linear differential operator**

$$LY(t) = X(t)$$

$$Y(0) = dY(0)/dt = \dots = 0$$

***h(t)* : Impulse Response**

$$Lh(t) = \delta(t)$$

$$h(0) = \dots = 0$$

$$h(t) = L_t^{-1}\delta(t)$$

Response of Linear Systems

Response

$$Y(t) = L_t^{-1}X(t) = L_t^{-1}\int_{-\infty}^{+\infty} X(\tau)\delta(t-\tau)d\tau = \int_{-\infty}^{+\infty} X(\tau)L_t^{-1}\delta(t-\tau)d\tau$$

X(t) = h(t) = 0 for t < 0

$$Y(t) = \int_{-\infty}^{+\infty} h(t-\tau)X(\tau)d\tau$$

$$Y(t) = \int_{-\infty}^{+\infty} h(\tau)X(t-\tau)d\tau$$

$$Y(t) = \int_0^t h(t-\tau)X(\tau)d\tau$$

Stationary Response Analysis

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Systems Function

$$H(\omega) = H(i\omega) = \int_{-\infty}^{+\infty} h(t) e^{-i\omega t} dt$$

$$Y(t) = \int_{-\infty}^{+\infty} h(t-\tau) X(\tau) d\tau = \int_{-\infty}^{+\infty} h(\tau) X(t-\tau) d\tau$$

h(t) = 0 for t < 0

$$Y(t) = \int_{-\infty}^{t} h(t-\tau) X(\tau) d\tau = \int_{0}^{+\infty} h(\tau) X(t-\tau) d\tau$$

Stationary Response Analysis

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Response Mean

$$E\{Y(t)\} = \int_{-\infty}^{+\infty} h(\tau) E\{X(t-\varepsilon)\} d\tau = \eta_X \int_{-\infty}^{+\infty} h(\tau) d\tau = \eta_X H(0)$$

Second Order Statistics

$$R_{YX}(\tau) = \int_{-\infty}^{+\infty} R_{XX}(\tau-\alpha) h(\alpha) d\alpha = R_{XX}(\tau) * h(\tau)$$

$$R_{YY}(\tau) = \int_{-\infty}^{+\infty} R_{YX}(\tau+\alpha) h(\alpha) d\alpha = \int_{-\infty}^{+\infty} R_{YX}(\tau-\alpha) h(-\alpha) d\alpha$$

Stationary Response Analysis

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$$R_{YY}(\tau) = R_{XY}(\tau) * h(\tau)$$

$$R_{XY}(\tau) = R_{XX}(\tau) * h(-\tau)$$

$$R_{YY}(\tau) = R_{YX}(\tau) * h(-\tau)$$

$$R_{YY}(\tau) = R_{XX}(\tau) * h(\tau) * h(-\tau)$$

System Identification

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For White Input

$$R_{XX}(\tau) = \delta(\tau) \qquad R_{YX}(\tau) = h(\tau)$$

Impulse Response

$$E\{Y(t+\tau)X(t)\} \approx \frac{1}{T}\int_0^t Y(t+\tau)X(t)dt \approx R_{YX}(\tau)$$

Spectral Response

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System Function

$$H^*(\omega) = \int_{-\infty}^{+\infty} h(-\tau) e^{-i\omega t} d\tau = \int_{-\infty}^{+\infty} h(\tau) e^{i\omega t} d\tau$$

Impulse Response Function

$$h(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{i\omega t} H(\omega) d\omega$$

Spectral Response

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$$S_{YX}(\omega) = S_{XX}(\omega) H(\omega)$$

$$S_{YY}(\omega) = S_{YX}(\omega) H^*(\omega)$$

$$S_{YY}(\omega) = S_{XY}(\omega) H(\omega)$$

$$S_{XY}(\omega) = S_{XX}(\omega) H^*(\omega)$$

$$S_{YY}(\omega) = S_{XX}(\omega) |H(\omega)|^2$$

Spectral Relationships

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Given

$$a_n \frac{d^n Y}{dt^n} + a_{n-1} \frac{d^{n-1} Y}{dt^{n-1}} + \ldots + a_0 Y = X(t,)$$

System Function

$$H(\omega) = \frac{1}{a_n (i\omega)^n + \ldots + a_0}$$

$$\overline{Y}(\omega) = H(\omega) \overline{X}(\omega)$$

Expected Value

$$E\{Y\} = H(0) E\{X\} = E\{X\} / a_0$$

Power Spectrum

$$S_{YY}(\omega) = S_{XX}(\omega) |H(\omega)|^2$$

Example: Langevin's Equation

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Brownian Motion of a Particle

$$\frac{dV}{dt} + \beta V = n$$

$$E\{n\} = 0$$

$$S_{nn}(w) = \alpha$$

Response Power Spectrum

$$S_{VV}(\omega) = |H(\omega)|^2 S_{nn}(\omega)$$

$$H(\omega) = \frac{1}{i\omega + \beta}$$

Example: Langevin's Equation

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Response Power Spectrum

$$|H(\omega)|^2 = \frac{1}{\omega^2 + \beta^2} \Rightarrow S_{VV}(w) = \frac{\alpha}{\omega^2 + \beta^2}$$

Response Autocorrelation

$$R_{VV}(\tau) = \frac{\alpha}{2\beta} e^{-\beta|\tau|}$$

$$E\{V^2\} = \frac{\alpha}{2\beta}$$

$$E\{V\} = 0$$

Brownian Motion

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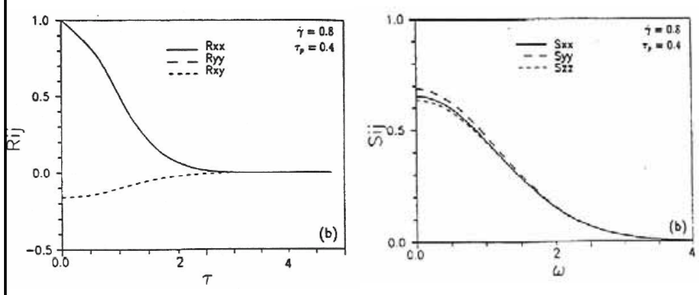

Autocorrelation and power spectrum of particle velocity

Base Isolation

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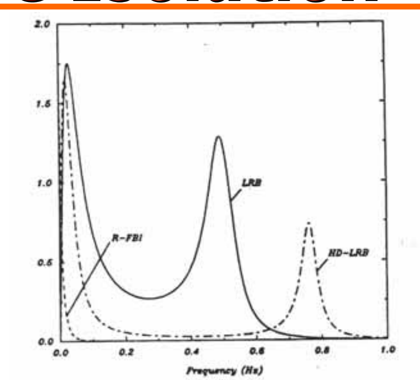

FIG. 6. Displacement Power Spectra for LRB, HD-LRB, and R-FBI Base-Isolation Systems for V_o = 42 m/s

Base Isolation

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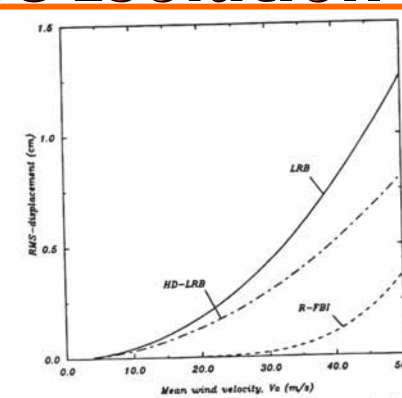

FIG. 9. RMS-Displacements for LRB, HD-LRB, and R-FBI Base-Isolation Syste