Clarkson University

Correlation and Power Spectrum of Stationary Processes

Definition: Autocorrelation of a stationary process $X(t)$ is defined as

$$R(\tau) = E\{X(t+\tau)X(t)\}.$$

For real processes, $R(\tau)$ is an even function. i.e.,

$$R(\tau) = R(-\tau).$$

Definition: Autocovariance of a WSS process $X(t)$ is defined as

$$C(\tau) = E\{(X(t)-\eta)(X(t+\tau)-\eta)\} = R(\tau)-\eta^2,$$

where

$$\eta = E\{X(t)\}.$$

Definition: Cross-Correlation of two jointly WSS processes $X(t)$ and $Y(t)$ is defined as

$$R_{XY}(\tau) = E\{X(t+\tau)Y(t)\} = R_{YX}(-\tau),$$

and their cross-covariance is given as

$$C_{XY}(\tau) = R_{XY}(\tau) - \eta_X \eta_Y = C_{YX}(-\tau).$$

If $Z(t) = aX(t) + bY(t)$ and $X(t)$ and $Y(t)$ are jointly WSS, then

$$R_{ZZ}(\tau) = a^2 R_{XX}(\tau) + ab(R_{XY}(\tau) + R_{YX}(\tau)) + b^2 R_{YY}(\tau).$$

Properties of Correlations

i) $R(0) \geq 0$.

ii) $R(\tau) \leq R(0)$.

iii) $R(\tau)$ is positive-definite. i.e., $\sum_i \sum_j a_i a_j^* R(\tau_i - \tau_j) \geq 0$

Proof:

$$R(0) = E\{(X(t))^2\} \geq 0.$$

Noting that

$$E\left\{\left[X(t+\tau) \pm X(t)\right]^2\right\} = 2\left[R(0) \pm R(\tau)\right] \geq 0,$$

it follows that $R(\tau) \leq R(0)$.

The third property follows form the following identity,

$$E\left\{\left|\sum_i a_i X(t_i)\right|^2\right\} = E\left\{\sum_i a_i X(t_i) \sum_j a_j^* X^*(t_j)\right\} = \sum_i \sum_j a_i a_j^* R(t_i - t_j) \geq 0.$$

Properties of Cross-Correlation

i) $R_{XY}^2(\tau) \leq R_{XX}(0) R_{YY}(0)$.
ii) $2R_{XY}(\tau) \leq R_{XX}(0) + R_{YY}(0)$.

The first property may be proved from the non-negativity of $E\left\{\left[X(t+\tau) + aY(t)\right]^2\right\}$ and the second follows from the geometric inequality.

Power Spectrum

Definition: The power spectrum (spectral density) of a WSS process $X(t)$ is defined as

$$S(\omega) = \int_{-\infty}^{+\infty} e^{-i\omega t} R(\tau) d\tau.$$

The Fourier inverse transform implies that

$$R(\tau) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{-i\omega\tau} S(\omega) d\omega.$$

Since $R(\tau) = R(-\tau)$ is an even function, $S(\omega)$ is also an even function of w, i.e. $S(\omega) = S(-\omega)$. Furthermore,

$$S(\omega) = \int_{-\infty}^{+\infty} R(\tau) \cos \omega\tau d\tau = 2\int_0^{\infty} R(\tau) \cos \omega\tau d\tau,$$

$$R(\tau) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} S(\omega) \cos \omega\tau d\omega = \frac{1}{\pi} \int_0^{\infty} S(\omega) \cos \omega\tau d\omega.$$

The variance of X is given as (with $\eta_X = 0$)

$$\sigma_X^2 = R(0) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} S(\omega) d\omega = \frac{1}{\pi}\int_0^{\infty} S(\omega) d\omega .$$

Cross Spectrum

Definition: The cross spectrum of jointly WSS processes $X(t)$ and $Y(t)$ is defined as

$$S_{XY}(\omega) = \int_{-\infty}^{+\infty} R_{XY}(\tau) e^{-i\omega\tau} d\tau .$$

Inversion formula

$$R_{XY}(\tau) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} S_{XY}(\omega) e^{i\omega\tau} d\tau .$$

For $\tau = 0$, it follows that

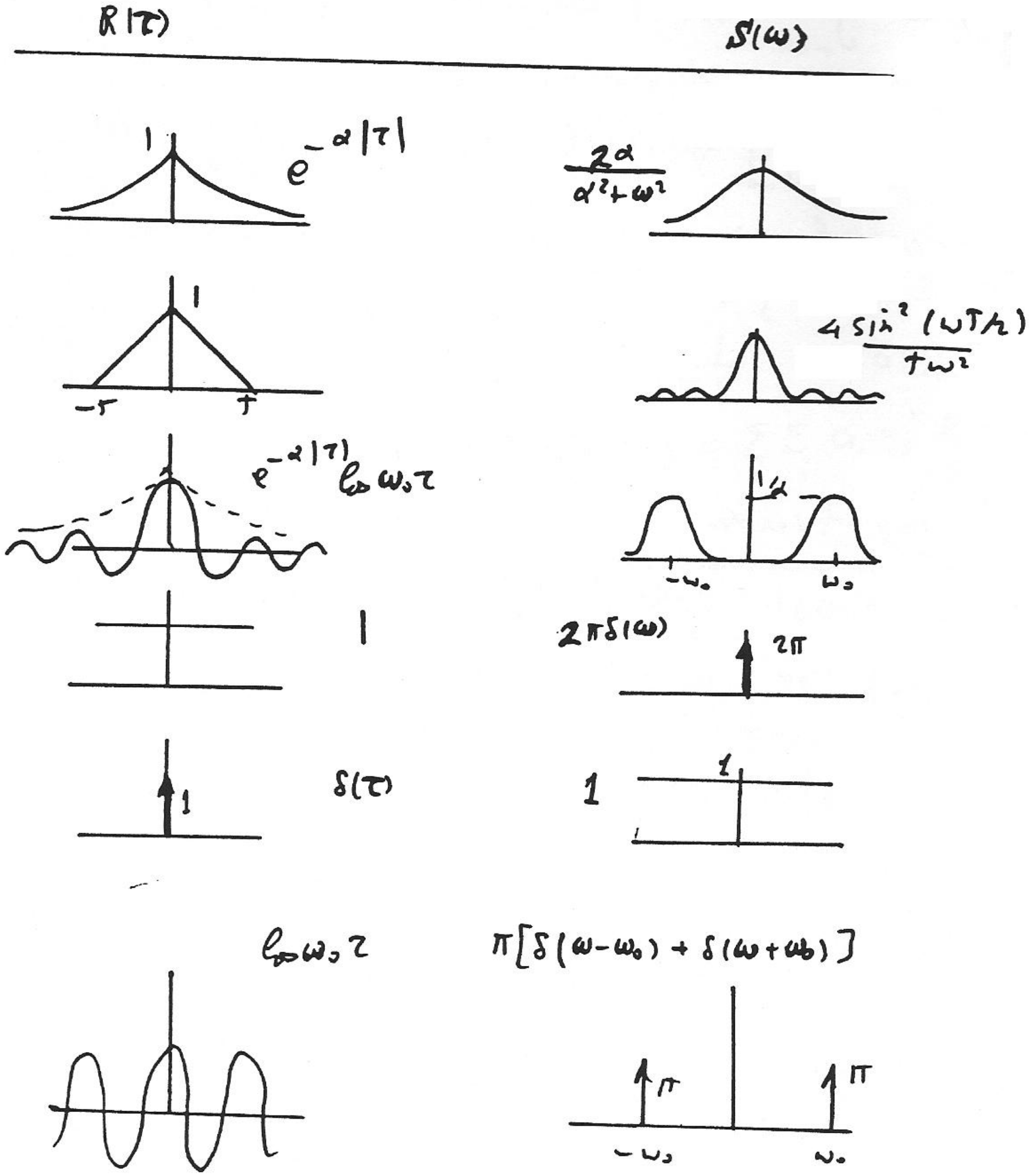
$$R_{XY}(0) = E\{X(t)Y(t)\} = \frac{1}{2\pi}\int_{-\infty}^{+\infty} S_{XY}(\omega) d\omega .$$

If $X(t)$ and $Y(t)$ are orthogonal processes, then

$$R_{XY}(\tau) = 0 ,$$

$$S_{XY}(\omega) = 0 .$$

$$\int_{-\alpha}^{+\alpha} e^{-i\omega\tau} R(\tau)\, d\tau$$



Particle Dispersion in Turbulent Flows

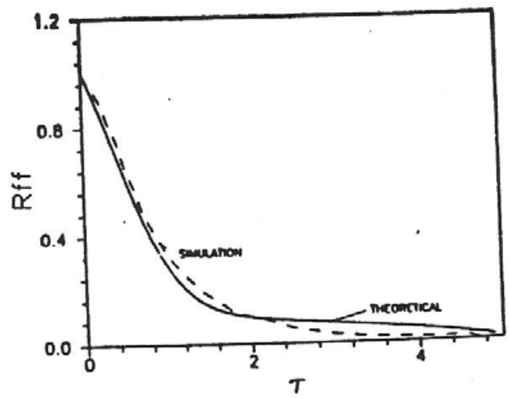


FIG. 1. Comparisons of Simulated and Theoretical Fluid Autocorrelations

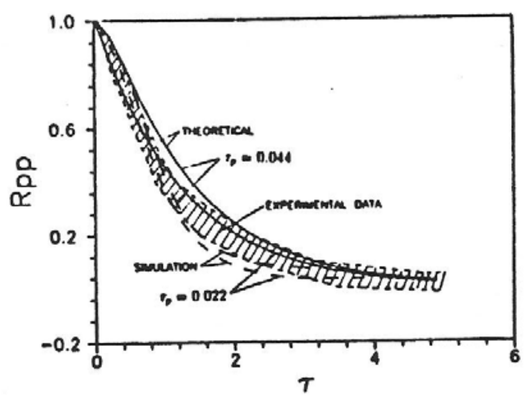


FIG. 2. Comparisons of Simulation and Theoretical Particle Autocorrelation Functions with Experimental Data of Snyder and Lumley

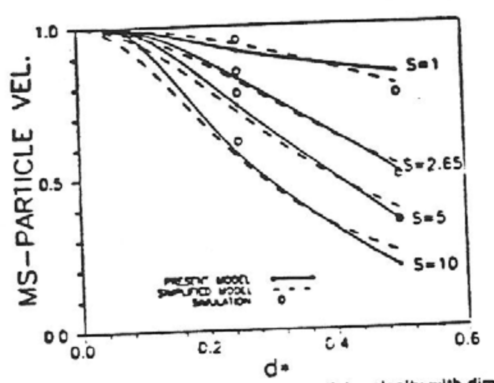


Fig. 3 Variations of relative mean-square particle velocity with dimensionless diameter *d*

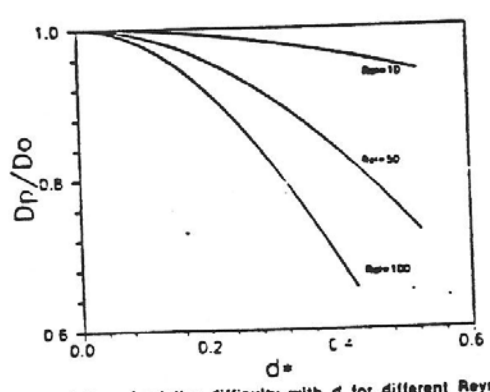


Fig. 5 Variation of relative difficulty with *d* for different Reynolds number Rel