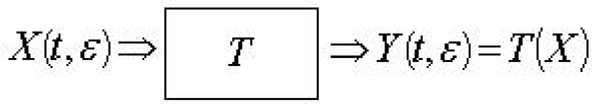
Transformation of Stochastic Processes

Consider a system T that transforms a stochastic process $X(t,\varepsilon)$ into another process $T(t,\varepsilon)$, i.e.

$$Y(t,\varepsilon) = T[X(t,\varepsilon)].$$



The system is deterministic if T only operates on t. The system is stochastic if T operates on both t and ε. That is, if $X(t,\varepsilon_1) = X(t,\varepsilon_2)$, then $Y(t,\varepsilon_1) = Y(t,\varepsilon_2)$ for a deterministic system. For a stochastic system the response to identical inputs are generally different.

Memory-less Systems

A system is called memory-less if

$$Y(t) = g[X(t)],$$

with g being a function of X. The response at time t then depends only on X(t). Therefore, the random variable Y(t) is an algebraic function of X(t). Thus, the first order density of Y(t) is given as

$$f_Y(y,t) = \sum_j \frac{f_X(x_j;t)}{|g'|}, \; x_j = g_j^{-1}(y).$$

Similarly, the joint density of $Y(t_1)$ and $Y(t_2)$ may be found from that of $X(t_1)$ and $X(t_2)$. The mean and autocorrelation of Y(t) are also given as

$$E\{Y(t)\} = \int_{-\infty}^{+\infty} g(x) f_X(x;t) dx,$$

$$E\{Y(t_1)Y(t_2)\} = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} g(x_1) g(x_2) f_X(x_1,x_2;t_1,t_2) dx_1 dx_2 .$$

Derivative of a Random Process

Let X(t) be differentiable with its derivative given by

$$X'(t) = \frac{dX}{dt} = \lim_{\varepsilon\to 0} \frac{X(t+\varepsilon) - X(t)}{\varepsilon} .$$

We assume that the limit exists in mean-square sense.

To find the mean and autocorrelation of X'(t), we proceed as follows:

Mean of X'(t):

$$E\left\{\frac{dX}{dt}\right\} = \frac{d}{dt} E\{X\} = \frac{d\eta(t)}{dt}.$$

Autocorrelation of X'(t):

$$R_{X'X'}(t_1, t_2) = E\{X'(t_1) X'(t_2)\} = E\left\{\frac{dX(t_1)}{dt_1} \quad \frac{dX(t_2)}{dt_2}\right\},$$

or

$$R_{X'X'}(t_1, t_2) = \frac{\partial^2 R_{XX}(t_1, t_2)}{\partial t_1 \partial t_2}.$$

Similarly cross correlation of X and X' may be found, i.e.

$$R_{XX'}(t_1, t_2) = \frac{\partial R_{XX}(t_1, t_2)}{\partial \partial t_2}.$$

If X(t) is a stationary process, then

$$R_{XX}(t_1, t_2) = R_{XX}(t_1 - t_2).$$

Thus,

$$R_{XX'}(\tau) = -\frac{dR_{XX}(\tau)}{d\tau}, \qquad \tau = t_1 - t_2,$$

$$R_{X'X'}(\tau) = -\frac{d^2 R_{XX}(\tau)}{d\tau^2}.$$

The mean-square of X' is then given as

$$E\left\{[X'(t)]^2\right\} = R_{X'X'}(0) = -\frac{d^2 R_{XX}(0)}{d\tau^2}.$$

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Random Linear Differential Equations

Consider a linear differential equation with random excitation of the form

$$L_t Y(t) = a_n \frac{d^n Y}{dt^n} + a_{n-1} \frac{d^{n-1} Y}{dt^{n-1}} + \dots + a_0 Y(t) = X(t), \tag{1}$$

where a_j are constants and X(t) is a random process. The initial conditions are

$$Y(0) = \frac{dY(0)}{dt} = \dots = \frac{d^{n-1} Y(0)}{dt^{n-1}} = 0 . \tag{2}$$

Mean of Y

To find $\eta_Y(t) = E\{Y(t)\}$ take the expected value of the differential equation and the initial conditions. The results are

$$L_t \eta_Y(t) = a_n \frac{d^n \eta_Y}{dt^n} + \dots + a_0 \eta_Y(t) = \eta_X(t), \tag{3}$$

with

$$\eta_Y(0) \frac{d\eta_Y(0)}{dt} = \dots = \frac{d^{n-1} \eta_Y(0)}{dt^{n-1}} = 0 . \tag{4}$$

Equation (3) is a deterministic equation for determining $\eta_Y(t)$.

Correlation of Y

Write Equation (1) at time t_2 and multiply by $X(t_1)$. i.e.,

$$X(t_1)\left[L_{t_2} Y(t_2) = X(t_2)\right] \tag{5}$$

Taking the expected value of (5), we find

$$L_{t_2} R_{XY}(t_1, t_2) = R_{XX}(t_1, t_2), \tag{6}$$

or

$$a_n \frac{\partial^n R_{XY}(t_1, t_2)}{\partial t_2^n} + \dots + a_0 R_{XY}(t_1, t_2) = R_{XX}(t_1, t_2) . \tag{7}$$

Multiplying the initial conditions given by Equation (2) by $X(t_1)$ and taking expected value we find

$$R_{XY}(t_1,0)=\frac{\partial R_{XY}(t_1,0)}{\partial t_2}=\ldots=\frac{\partial^{n-1}R_{XY}(t_1,0)}{\partial t_2^{n-1}}=0\,. \tag{8}$$

Solution of the deterministic equation given by (7) with initial conditions given by (8) yields $R_{XY}(t_1, t_2)$.

Now write Equation (1) at time t_1 and multiply the result by $Y(t_2)$. Taking the expected value we find

$$L_t \;= R_{YY}(t_1,t_2)=R_{XY}(t_1,t_2), \tag{9}$$

or

$$a_n\frac{\partial^n R_{YY}(t_1,t_2)}{\partial t_1^n}+\ldots+a_0 R_{YY}(t_1,t_2)=R_{XY}(t_1,t_2). \tag{10}$$

The solution to (10) with initial conditions

$$R_{YY}(0,t_2)=\frac{\partial R_{YY}(0,t_2)}{\partial t_1}=\ldots=\frac{\partial^{n-1}R_{YY}(0,t_2)}{\partial t_1^{n-1}}=0\,, \tag{11}$$

Gives $R_{YY}(t_1,t_2)$.

Note that even if X(t) is a stationary process, Y(t) is a non-stationary process. The reason is that at t = 0 the initial conditions given by (2) are forced on Y(t).