ME 529 - Stochastics

Transformation of Stochastic Processes

Goodarz Ahmadi

Department of Mechanical and Aeronautical Engineering
Clarkson University
Potsdam, NY 13699-5725

Stochastic Transformation

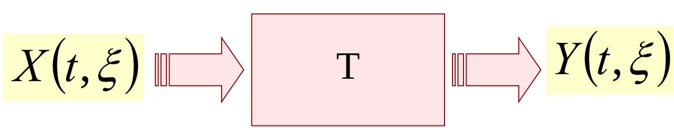
Outline

- Memory-less Systems
- Derivative of Random Processes
- Mean and Autocorrelation
- Random Linear Differential Equations
- Evaluation of Mean and autocorrelation of Response

Memory-less Systems

Transformation of Stochastic Process

$$Y(t,\xi) = T[X(t,\xi)]$$



Memory-less Systems

System is deterministic if T operates on t.

⇒ **if** $Y(t,\xi_1) = Y(t,\xi_2)$ **then** $X(t,\xi_1) = X(t,\xi_2)$

System is stochastic if T operates on t and ξ.

⇒ **responses to identical inputs differ.**

Memory-less Systems

Clarkson University

$$Y(t) = g[X(t)]$$

1st Order Density

$$f_Y(y;t) = \sum_j \frac{f_X(x_j;t)}{|g'|} \qquad x_j = g_j^{-1}(y)$$

Mean

$$E\{Y(t)\} = \int_{-\infty}^{+\infty} g(x) f_X(x;t)\,dx$$

Autocorrelation

$$E\{Y(t_1)Y(t_2)\} = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} g(x_1)g(x_2) f_X(x_1,x_2;t_1,t_2)\,dx_1 dx_2$$

Derivative of a Random Process

Clarkson University

Time Derivative

$$X'(t) = \frac{dX}{dt} = \lim_{\varepsilon\to 0}\frac{X(t+\varepsilon) - X(t)}{\varepsilon}$$

Mean

$$E\left\{\frac{dX}{dt}\right\} = \frac{d}{dt}E\{X\} = \frac{d}{dt}\eta(t)$$

Autocorrelation

$$R_{X'X'}(t_1,t_2) = E\{X'(t_1)X'(t_2)\} = E\left\{\frac{dX(t_1)}{dt_1}\frac{dX(t_2)}{dt_2}\right\}$$

or

$$R_{X'X'}(t_1,t_2) = \frac{\partial^2 R_{XX}(t_1,t_2)}{\partial t_1 \partial t_2}$$

Cross Correlation X & X'

$$R_{XX'}(t_1,t_2) = \frac{\partial}{\partial t^2} R_{XX}(t_1,t_2)$$

Derivative of a Random Process

Clarkson University

For X(t) stationary

$$R_{XX}(t_1,t_2) = R_{XX}(t_1 - t_2)$$

$$R_{XX'}(\tau) = -\frac{dR_{XX}(\tau)}{d\tau} \qquad \tau = t_1 - t_2$$

$$R_{X'X'}(\tau) = -\frac{d^2 R_{XX}(\tau)}{d\tau^2}$$

Mean Square

$$E\{[X'(t)]^2\} = R_{X'X'}(0) = -\frac{d^2 R_{XX}(0)}{d\tau^2}$$

Random Linear Differential Equations

Clarkson University

$$L_t Y(t) = a_n \frac{d^n Y}{dt^n} + a_{n-1}\frac{d^{n-1}Y}{dt^{n-1}} + \ldots + a_0 Y(t) = X(t)$$

$$Y(0) = \frac{dY(0)}{dt} = \ldots = \frac{d^{n-1}Y(0)}{dt^{n-1}} = 0$$

Mean of Y

$$\eta_Y(t) = E\{Y(t)\}$$

Taking expected value of diff eqn and I.C.'s

$$L_t \eta_Y(t) = a_n \frac{d^n \eta_Y}{dt^n} + \ldots + a_0 \eta_Y(t) = \eta_X(t)$$

$$\eta_Y(0) = \frac{d\eta_Y(0)}{dt} = \ldots = \frac{d^{n-1}\eta_Y(0)}{dt^{n-1}} = 0$$

Random Linear Differential Equations

Clarkson University

Cross Correlation of Y and X

$$X(t_1)\left[L_{t_2}Y(t_2)=X(t_2)\right]$$

$$L_{t_2}R_{XY}(t_1,t_2)=R_{XX}(t_1,t_2)$$

or

$$a_n\frac{\partial^n R_{XY}(t_1,t_2)}{\partial t_2^n}+\ldots+a_0R_{XY}(t_1,t_2)=R_{XX}(t_1,t_2)$$

Multiply ICs by $X(t_1)$ & taking expected value:

$$R_{XY}(t_1,0)=\frac{\partial R_{XY}(t_1,0)}{\partial t_2}=\ldots=\frac{\partial^{n-1}R_{XY}(t_1,0)}{\partial t_2^{n-1}}=0$$

Random Linear Differential Equations

Clarkson University

Write $X(t_1)$, multiply $Y(t_2)$, take expected value

$$L_{t_1}R_{YY}(t_1,t_2)=R_{XY}(t_1,t_2)$$

or

$$a_n\frac{\partial^n R_{YY}(t_1,t_2)}{\partial t_1^n}+\ldots+a_0R_{YY}(t_1,t_2)=R_{XY}(t_1,t_2)$$

$$R_{YY}(0,t_2)=\frac{\partial R_{YY}(0,t_2)}{\partial t_1}=\ldots=\frac{\partial^{n-1}R_{YY}(0,t_2)}{\partial t_1^{n-1}}=0$$

Note: Y(t) is non-stationary

Stochastic Transformation

Clarkson University

Concluding Remarks

- Memory-less Systems
- Derivative of a Random Process
- Statistics of Derivative
- Random Linear Differential Equations
- Response Mean and Autocorrelation

Stochastic Transformation

Clarkson University

Thank you!

Questions?